

Contributor Profile: Walther Janous



Walther Janous was born on March 16th, 1953 in Innsbruck, Tyrol (the “heart of the Alps”). His parents, a chemist and a now-retired internist, had to build a new existence when Walther and his brother were young. (This was largely due to World War II and the difficulties immediately following.)

While in secondary school, the two brothers were among the “pioneers” in Austria of the then-starting Math Olympiad. After graduating in 1971, Walther attended university in Innsbruck. He enrolled in mathematics, information technology, philosophy, psychology, and educational sciences; he also worked as a Teaching Assistant from third year on. Having finished all the courses and the two required theses, he had the opportunity for one year of stay at Ohio State University in Columbus, where he was fortunate enough to meet many impressive mathematicians, the most remarkable being Erdős.

In 1978, he started teaching mathematics and philosophy at secondary schools in Innsbruck. For the past 28 years, he has been working for the Ursulinengymnasium, where he looks after girls aged 10 to 18, and prepares them for the “Matura”, the Austrian school-exit certificate.

Walther has been involved for almost 30 years with the Austrian Math Olympiad program, serving as a coach and submitting problem proposals. He was deputy-leader of the Austrian IMO team in Slovenia, and he will be the leader of the Austrian team participating in the inaugural Middle European Math Olympiad (MEMO) this year. He is also responsible for the Problem Section of the *Wissenschaftliche Nachrichten* (WN), a magazine for Austrian science teachers published three times a year. He has been an occasional referee for journals such as *CRUX with MAYHEM*, JIPAM, AML, and WN, mostly for papers on inequalities. *CRUX with MAYHEM* readers will no doubt know of Walther as one of its most prolific problem solvers. Indeed, he has been solving and posing problems for our readers since 1984.

Walther has been married since 1995. His wife Marlies teaches German and French in secondary school. His hobbies include reading, going to concerts with all kinds of “classic” music (from Baroque to the 21st century), mountaineering, and doing mathematics, of course.

One of his beliefs is that we all have to strive for the implementation of more fairness all around our only world—this includes basically education, education, education, . . . for as many persons as possible. This probably would help to reduce violence and aggression and to build self-confidence and self-esteem as central components of a human life!

SKOLIAD No. 105

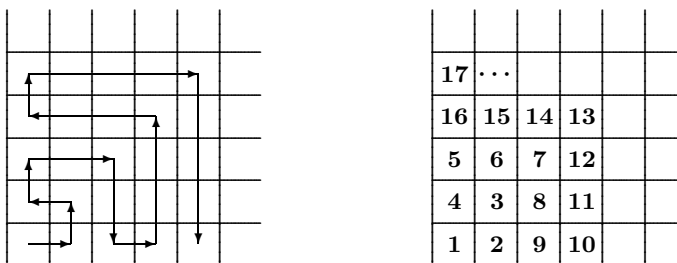
Robert Bilinski

Please send your solutions to the problems in this edition by **April 1, 2008**. A copy of **MATHEMATICAL MAYHEM Vol. 7** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our contest this month is the National Bank of New Zealand Competition 2004. Our thanks go to Warren Palmer, Otago University, Otago, New Zealand for providing us with this contest and for permission to publish it.

National Bank of New Zealand Junior Mathematics Competition 2004 (Years 9 and above) 1 hour allowed

1 (For year 9 only). Linda starts to write down the natural numbers in the square cells of a very large piece of graph paper. (The graph paper is much larger than shown below.) She starts at the bottom left corner and writes down the numbers using the following arrangement:



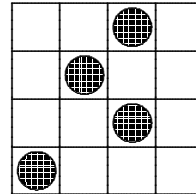
(The arrangement is suggested in the left diagram; some of the numbers are shown in the right diagram.)

We identify each of the cells using co-ordinates (a, b) , where a is the number of positions to the right, and b is the number of positions up from the bottom. For example, the cell containing the number 1 has the co-ordinates $(1, 1)$, while the cell containing the number 8 has the co-ordinates $(3, 2)$.

- What are the co-ordinates of the cell containing the number 15?
- Starting with 1, 9, ..., every second number along the bottom row follows a certain pattern. In a few words, or using an algebraic expression, describe these numbers.

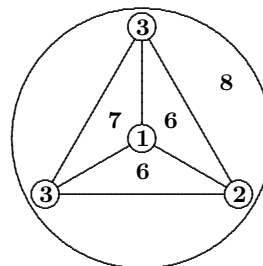
- (c) The cell containing the number 21 has the co-ordinates (5, 5). What is the number contained in the cell with co-ordinates (6, 6)? As well, find the number contained in the cell with co-ordinates (7, 7).
- (d) What is the number contained in the cell with co-ordinates (20, 20)?
- (e) What are the coordinates of the cell containing the number 2004?

2. The diagram shows a 4×4 grid containing four coins. Imagine that we have enough coins available to place anywhere we like on the grid. However, we would like to place coins so that we do not have three placed anywhere along a line, either horizontally, or vertically, or diagonally.



- (a) Imagine that we add one more coin to the given layout. In how many different squares could we place the extra coin so that we would not have three coins placed anywhere along a line?
- (b) Is it possible to add two more coins into the given layout so that we would not have three coins placed anywhere along a line? If it is possible, show by drawing a diagram where the two extra coins could be placed. If it is not possible, explain why not.
- (c) Imagine now that the grid contains no coins at all. What is the smallest number of coins which could be placed onto the grid so that we would not have three placed anywhere along a line, but if we were then to add an extra coin we could not avoid having three placed along a line? Describe, perhaps including a diagram, where the coins would be placed.
- (d) Imagine again that the grid contains no coins at all. What is the largest number of coins which could be placed onto the grid so that nowhere are there three coins placed anywhere along a line? Describe, perhaps including a diagram, where the coins would be placed.

3. The diagram shows an equilateral triangle divided into three smaller triangles. Small circles have been placed on each of the vertices, and positive whole numbers (in this case 3, 3, 2, and 1) have been written inside each small circle. Overall the shape forms four regions: three triangles and an outer region. In each of these regions the sum of the corresponding vertices has been written. For example, the outer region contains the value 8, because $\textcircled{3} + \textcircled{3} + \textcircled{2} = 8$.



In this question we shall be investigating what happens when the numbers in the small circles are changed. (Throughout this question, only positive whole numbers will be used.)

- (a) Imagine that the number in each one of the small circles is ⑤. What is the total when the numbers inside all four regions are added together?
- (b) Find possible numbers inside each of the four small circles so that the sums in the three triangles are 8, 9, and 10, respectively, while the sum of the outer region is 6.
- (c) Find possible numbers inside each of the four small circles so that the sums in the three triangles are 8, 9, and 9, respectively. (Do not worry about the sum of the outer region in this part of the question.)
- (d) Is your answer to part (c) the only possible answer? If it is, explain why no other answer is possible. If it is not, find another answer.
- (e) We have been using positive whole numbers throughout this question. In a few words, or using an algebraic expression, give a general description of the total when the numbers inside all four regions are added together. Explain your reasoning for your description.

4. A class of students votes to select one candidate as their representative on the school council. Their teacher decides on the following voting system:

“You have to rank the three candidates in order: first, second, and third. Your first choice will receive one point; your second choice will receive two points, and your third choice will receive four points. The winner will be the student with the smallest total.”

After the voting has been completed, the teacher discovers that there is a problem with this voting system. She explains the problem to the principal:

“The student with the smallest score is Diane, who received 44 points. However, only four people voted for her as their first choice. Next was Belinda with 45 points. She received more first choice votes than anyone else. Colin was in last place with 51 points, and he had more people voting for him as their third choice than voted for the other candidates. It looks as though I will have to announce to the class that Diane is the winner, even though she had the smallest number of people voting for her as first choice.”

- (a) Show that 20 students took part in the voting.
- (b) How many people voted for Belinda as their first choice?
- (c) Explain why your answer to (b) is the only possible answer which fits the teacher’s description of how the votes were cast.
- (d) How many people voted for Belinda as their second choice, and how many people voted for her as their third choice?

5. Ari has cut some regular pentagons out of cardboard and is joining them together to make a ring (see Figure 1). He has cut them using a template so that they are all the same size.

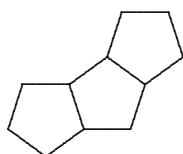


Figure 1

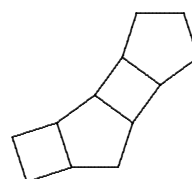


Figure 2

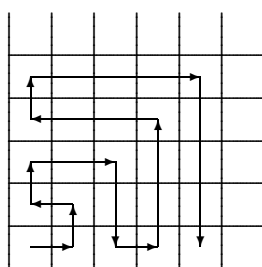
- (a) The external angle of a regular pentagon is 72° . Explain how this value is calculated.
- (b) When the ring is complete, how many pentagons will there be?

Next Ari decides to join his pentagons with squares which have the same side length (see Figure 2). He would like to combine them all together to make a new ring with alternating squares and pentagons.

- (c) Is it possible for Ari to construct a ring in this way? If it is possible, explain why. If it is not possible, explain why not.
- (d) Ari finally decides to construct a ring using regular hexagons (six sides) joined together. (This is not shown in any diagram.) If the hexagons have side length of exactly one unit, what is the area of the shape enclosed inside the ring?

Compétition 2004 junior de mathématiques de la Banque Nationale de Nouvelle-Zélande (secondaire 3 et plus) 1 heure au total

1 (Pour les 9^{ième} années ou secondaire 3 seulement). Linda commence à écrire les nombres naturels dans les carrés d'une très grande feuille de papier quadrillé. (Le papier est beaucoup plus grand qu'indiqué en bas.) Elle commence en bas à gauche et écrit les nombres en utilisant l'arrangement :



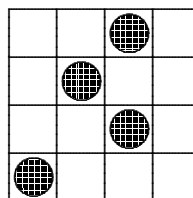
17	...			
16	15	14	13	
5	6	7	12	
4	3	8	11	
1	2	9	10	

(L'arrangement est suggéré par le diagramme à gauche ; quelques nombres sont placés dans le diagramme de droite.)

On identifie les cases en utilisant des coordonnées (a, b) , où a est le nombre de positions à la droite, et b est le nombre de position vers le haut à partir du bas. Par exemple, la case contenant 1 a pour coordonnées $(1, 1)$, alors que celle contenant 8 a les coordonnées $(3, 2)$.

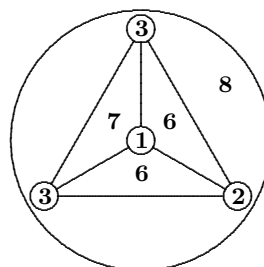
- (a) Quelles sont les coordonnées de la case contenant 15 ?
- (b) En commençant avec 1, 9, ..., chaque deuxième nombre le long de la ligne du bas suit un patron. En quelques mots ou en utilisant une expression algébrique, décrivez ces nombres.
- (c) La case contenant le nombre 21 a pour coordonnées (5, 5). Quel est le nombre contenu dans la case (6, 6) ? De même, trouver le nombre dans la case de coordonnées (7, 7).
- (d) Quel est le nombre dans la case de coordonnées (20, 20) ?
- (e) Quelles sont les coordonnées de la case contenant 2004 ?

2. La figure montre un quadrillage 4×4 contenant quatre jetons. Imaginez qu'il y a assez de jetons pour les placer où nous voulons sur le quadrillage. Cependant, nous voulons placer les jetons pour qu'il n'y en ait pas trois par ligne, que ce soit horizontalement, verticalement ou diagonalement.



- (a) Imaginez que nous ajoutons un jeton à la disposition actuelle. Dans combien de carrés pouvons nous placer ce jeton de plus pour ne pas avoir trois jetons sur une même ligne ?
- (b) Est-il possible d'ajouter deux jetons de plus à la disposition actuelle pour ne pas avoir trois jetons sur une même ligne ? Si c'est possible, montrer où les placer en faisant un dessin. Sinon, expliquer pourquoi ce n'est pas possible.
- (c) Imaginez que le quadrillage n'ait plus de jetons. Quel est le plus petit nombre de jetons que l'on peut placer sur le quadrillage pour ne pas en avoir trois sur la même ligne ? Décrire, avec possiblement un dessin à l'appui, où les jetons seraient placés.
- (d) Imaginez de nouveau un quadrillage vide. Quel est le plus grand nombre de jetons que l'on peut placer sur le quadrillage pour ne pas en avoir trois sur la même ligne ? Décrire, avec possiblement un dessin à l'appui, où les jetons seraient placés.

3. On retrouve sur le dessin un triangle équilatéral divisé en trois plus petits triangles. Les petits cercles ont été placés sur les sommets, et des nombres entiers positifs (ici 3, 3, 2 et 1) ont été écrits dans ces petits cercles. Globalement, la forme crée quatre régions : trois triangles et une région extérieure. On écrit dans chacune de ces régions la somme des sommets correspondants. Par exemple, dans la région extérieure, on a écrit la valeur 8, car $\textcircled{3} + \textcircled{3} + \textcircled{2} = 8$.



Dans cette question, nous allons explorer ce qui se passe lorsque les nombres dans les petits cercles sont changés. (Seulement les nombres entiers positifs seront utilisés.)

- (a) Imaginez que le nombre dans chaque cercle est ⑤. Quel est le total des nombres écrits dans les quatre régions ?
- (b) Trouvez des nombres à mettre dans les cercles pour que les sommes dans les triangles soient de 8, 9 et 10 alors que dans la région extérieure, elle soit de 6.
- (c) Trouvez des nombres à mettre dans les cercles pour que les sommes dans les triangles soient de 8, 9 et 9 (ne vous préoccupez pas de la somme dans la région extérieure dans cette sous-question).
- (d) Votre réponse au (c) est-elle la seule possible ? Si oui, expliquer pourquoi elle est la seule, sinon, trouver en d'autres.
- (e) Nous avons utilisé des entiers positifs dans cette question. En quelques mots, ou en utilisant des expressions algébriques, donner une description générale du total lorsque les nombres dans les quatre régions sont additionnés ensemble. Expliquez le raisonnement derrière votre description.

4. Une classe d'étudiants vote pour sélectionner un candidat comme représentant sur le conseil d'établissement. Leur professeur décide d'utiliser le système de votation suivant :

«Vous devez ordonner les trois candidats dans l'ordre : le premier, le deuxième et le troisième ; votre premier choix obtient un point ; votre deuxième en reçoit deux points, et votre troisième en reçoit quatre points. Le gagnant est l'étudiant avec le plus petit total.»

Après que le scrutin soit fini, le professeur découvre qu'il y a un problème avec ce système de votation. Elle explique le problème au directeur :

«L'étudiant avec le score le plus bas est Diane, qui a reçu 44 points. Par contre, seulement quatre personnes ont voté pour elle en première place. Ensuite, on retrouve Belinda avec 45 points. Elle a reçu le plus de première place. Colin est en dernier avec 51 points, et il a plus de gens votant pour lui en troisième que les autres candidats. Il semblerait que je dois prononcer Diane gagnante, même si elle a reçu le moins de premières places.»

- (a) Montrer que 20 étudiants ont voté.
- (b) Combien de gens ont voté pour Belinda en première place ?
- (c) Expliquez pourquoi votre réponse au (b) est la seule réponse possible qui correspond à la description du déroulement du vote de la professeur.
- (d) Combien de personnes ont voté pour Belinda dans leur deuxième choix, et combien ont voté pour elle dans leur troisième choix ?

5. Ari a coupé des pentagones réguliers dans des cartes et les met ensemble pour faire un anneau (voir Figure 1). Il les a coupés à l'aide d'un patron pour qu'ils soient de la même grandeur.

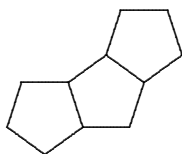


Figure 1

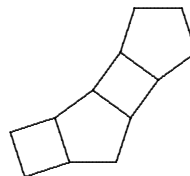


Figure 2

- (a) L'angle extérieur d'un pentagone régulier est 72° . Expliquez comment on calcule cette valeur.
- (b) Quand l'anneau est fini, combien de pentagones y aura-t-il ?

Ensuite, Ari décide d'ajouter des carrés de même longueur de côté (voir Figure 2). Il voudrait alors faire un nouvel anneau en alternant les carrés et les pentagones.

- (c) Est-il possible pour Ari de construire un anneau de cette manière ? Si c'est possible, expliquez pourquoi. Sinon, expliquez pourquoi pas.
- (d) Ari décide finalement de construire un anneau utilisant des hexagones réguliers (six côtés). (Il n'y a pas de dessin.) Si les côtés des hexagones ont une longueur d'une unité exactement, quelle est l'aire de la forme incluse à l'intérieur de l'anneau ?

Ensuite, nous donnons les solutions du concours de l'Association Mathématique du Québec 2005 [2007 : 66–69].

1. (Le robot et les pommes.) Une caisse de bois est séparée en 9 compartiments comme indiqué sur le dessin. Un ingénieur a programmé un robot pour qu'il remplisse la caisse de pommes par paquets de quatre en laissant tomber une pomme dans chaque compartiment de façon à former un carré 2×2 .

6	16	10
11	28	17
5	12	7

Est-il possible pour le robot d'aboutir à la configuration à droite à partir d'une caisse vide ?

Solution officielle.

La réponse est oui. On applique l'opération six fois au carré 2×2 du coin Nord-Ouest, cinq fois au carré 2×2 du coin Sud-Ouest, sept fois au carré 2×2 du coin Sud-Est et enfin dix fois au carré 2×2 du coin Nord-Est.

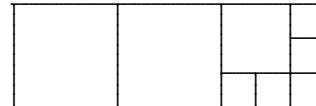
Aussi solutionné par Jochem van Gaalen, étudiant, Medway High School, Ilderton, ON.

2. (Eight squares in a rectangle.) Divide a rectangle of length 9 cm and width 3 cm into eight squares.

Solution by Jochem van Gaalen, student, Medway High School, Ilderton, ON, with a diagram from the official solution.

The table to the right shows all the possible side lengths for a square and the corresponding area of the square. If we use five 1×1 squares, one 2×2 square and two 3×3 squares, we have a total area of 27 (as does the rectangle) and we have exactly 8 squares.

Side length	1	2	3
Area	1	4	9



3. (Une étonnante distribution.) Une distribution statistique est composée de 10 nombres naturels : $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Lorsqu'ils sont placés en ordre croissant, ces nombres nous donnent en fait la distribution suivante : $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$. Nous avons plusieurs informations :

- (1) Les couples $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ et (x_5, y_5) , sont tous sur la droite d d'équation $y = -2x + 24$.
- (2) La moyenne de cette distribution est 9,4.
- (3) La médiane et le mode ont tous deux la même valeur.
- (4) Les nombres x_3 et x_4 sont consécutifs.
- (5) Le premier membre de la distribution vaut 1.
- (6) La droite d croise la parabole d'équation $y = \frac{1}{2}x^2 + 8x - 8$ au point (x_2, y_2) .

Trouver les valeurs de la distribution originale $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Suggestion : La médiane est le nombre tel que 50% des observations sont plus petites ou égales à ce nombre et 50% supérieures ou égales. Le mode est la valeur qui est observée le plus souvent.

Solution officielle, modifiée par le rédacteur.

À cause de (5), $x_1 = 1$. Par (1), le couple (x_1, y_1) est sur la droite d d'équation $y = -2x + 24$ et nous trouvons $y_1 = 22$. Il suffit d'utiliser (6) pour trouver le point (x_2, y_2) . En effet, nous avons les deux équations $y_2 = -\frac{1}{2}x_2^2 + 8x_2 - 8$ et $y_2 = -2x_2 + 24$. Il y a deux solutions possibles pour x_2 , soit $x_2 = 16$ et $x_2 = 4$. La solution $x_2 = 16$ est à rejeter car elle donne $y_2 = -8$, ce qui n'est pas admis. Il reste $x_2 = 4$ et $y_2 = -8 + 24 = 16$.

Soit m la médiane. Elle doit être égale au mode, par (3), ce qui implique que m est un nombre de la distribution. En tenant compte de l'ordre croissant des nombres $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$, on obtient $m = \frac{1}{2}(x_5 + y_5)$. Si $x_5 < y_5$, on a $x_5 < m < y_5$. Mais ce n'est pas possible parce qu'il n'y a pas de nombres de la distribution entre x_5 et y_5 . Ainsi $x_5 = y_5$. En utilisant (1), on obtient $x_5 = -2x_5 + 24$, d'où $x_5 = 8$ et donc $y_5 = 8$.

À ce stade, notre distribution s'écrit $1, 4, x_3, x_4, 8, 22, 16, y_3, y_4, 8$. La moyenne de la distribution est

$$\begin{aligned} & \frac{x_1 + x_2 + x_3 + x_4 + x_5 + y_5 + y_4 + y_3 + y_2 + y_1}{10} \\ &= \frac{1 + 4 + x_3 + x_4 + 8 + 8 + (-2x_4 + 24) + (-2x_3 + 24) + 16 + 22}{10} \\ &= \frac{107 - x_3 - x_4}{10}. \end{aligned}$$

En tenant compte des faits que la moyenne est $9,4$ et $x_4 = x_3 + 1$, on obtient $-2x_3 + 106 = 94$; ainsi, $x_3 = 6$ et $x_4 = 7$. On déduit finalement les valeurs $y_3 = 12$ et $y_4 = 10$. La distribution originale est donc

$$1, 4, 6, 7, 8, 22, 16, 12, 10, 8.$$

4. (La belle somme de Gilbert.) Considérons les 6 façons possibles de permuter (c'est-à-dire mélanger) les chiffres du nombre 123 et additionnons le tout. La somme trouvée s'écrit $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

Quel résultat aurions-nous obtenu si nous avions fait la somme des 5040 façons de permuter les chiffres du nombre 1234567 ?

Solution officielle.

Dans les 5040 permutations, chaque chiffre, parmi $1, 2, 3, 4, 5, 6, 7$, apparaît $5040/7 = 720$ fois comme unité, le même nombre de fois comme dizaine, comme centaine, etc. Le total est donc

$$\begin{aligned} & 720 \cdot (1 + 2 + 3 + 4 + 5 + 6 + 7) \\ & \quad \cdot (1 + 10 + 100 + 1000 + 10000 + 100000 + 1000000) \\ & = 720 \times 28 \times 1111111 = 22399997760. \end{aligned}$$

5. (Le voyage à Québec.) Juliette et Philippe partent en même temps et parcourent les 250 km qui séparent Montréal de Québec dans deux voitures identiques. Philippe parcourt la première moitié du trajet à 80 km/h et la seconde moitié à 120 km/h. En fait, il arrive en même temps que Juliette qui a roulé tout le long à vitesse constante. La consommation d'essence de ce type de voiture dépend de la vitesse du véhicule. Elle est donnée par la formule $c = 10 + \frac{v}{20}$, où v est la vitesse en km/h et c la consommation en litres par 100 km. Sachant que ce jour-là, le litre d'essence vaut 0,80\$, combien ont-ils dépensé ensemble pour le voyage?

Solution officielle.

Les consommations respectives de Philippe durant chacune des moitiés sont $c_1 = 10 + \frac{80}{20} = 14$ et $c_2 = 10 + \frac{120}{20} = 16$ (en litres par 100 km). Comme il parcourt durant chacune des moitiés 125 km, sa consommation totale est $1,25(c_1 + c_2) = 1,25 \times 30$. Puisqu'un litre vaut 0,80\$, le coût total de Philippe est $0,80 \times 1,25 \times 30 = 30\$$.

Calculons maintenant le coût total de Juliette. Il faut donc établir sa vitesse. Comme elle a parcouru le trajet dans le même temps que Philippe, sa vitesse est égale à la vitesse moyenne de Philippe. Contrairement à ce qu'on pourrait croire, la vitesse moyenne sur l'ensemble du parcours n'est pas égale à la moyenne des vitesses sur le parcours. La vitesse moyenne est donnée par $v = 250/(t_1 + t_2)$, où t_1 et t_2 représentent les temps de parcours (en heures) de chacune des moitiés. On a

$$t_1 = \frac{125 \text{ km}}{80 \text{ km/h}} = \frac{125}{80} \text{ h} \quad \text{et} \quad t_2 = \frac{125 \text{ km}}{120 \text{ km/h}} = \frac{125}{120} \text{ h}.$$

Donc,

$$v = \frac{250 \text{ km}}{\frac{125}{80} \text{ h} + \frac{125}{120} \text{ h}} = 96 \text{ km/h}.$$

La consommation de Juliette est de $10 + \frac{96}{20} = 14,8$ litres par 100 km. Sa consommation totale d'essence aura donc été de $14,8 \times 2,5$ et le coût total de Juliette est $0,8 \times 14,8 \times 2,5 = 29,60\$$.

Ensemble, le voyage aura coûté en carburant $30,00 + 29,60 = 59,60\$$.

Une solution erronée a été soumise.

6. (Les âges multiples.) Du 21 août 1989 au 7 mai (inclusivement) 1990, Jean a eu 5 fois l'âge de sa fille Claire. Du 8 mai au 20 août (inclusivement) 1992, Jean a eu 4 fois l'âge de sa fille. Trouver la date de naissance de chacun.

Note : par âge, on entend la définition usuelle qui est le nombre d'années complètes écoulées depuis la naissance.

Solution officielle, modifiée par le rédacteur.

On désignera par J l'année de naissance de Jean et par C l'année de naissance de Claire. Nous allons calculer sur deux dates, la première choisie arbitrairement entre le 21 août 1989 et le 7 mai 1990, soit le 31 décembre 1989, et l'autre choisie arbitrairement entre le 8 mai et le 20 août 1992, soit le 1er juillet 1992.

Il est clair que les anniversaires de Jean et Claire sont le 8 mai ou le 21 août, sans qu'on puisse préciser lequel est lequel. Nous avons deux cas à considérer.

Cas 1. Jean est né un 8 mai et Claire, un 21 août.

Le 31 décembre 1989, l'âge de Jean est $1989 - J$ et l'âge de Claire est $1989 - C$. Nous avons

$$1989 - J = 5(1989 - C). \quad (1)$$

Le 1er juillet 1992, l'âge de Jean est $1992 - J$ et l'âge de Claire est $1991 - C$ (pas $1992 - C$ parce que l'anniversaire de Claire n'est pas encore arrivé en 1992). Nous avons

$$1992 - J = 4(1991 - C). \quad (2)$$

En soustrayant (1) de (2), on trouve $3 = C - 1981$; donc $C = 1984$ et $J = 1964$.

En effet, le 31 décembre 1989, Jean avait 25 ans, soit cinq fois l'âge de Claire. Le 1er juillet 1992, Jean avait 28 ans, soit quatre fois l'âge de Claire.

Cas 2. Jean est né un 21 août et Claire, un 8 mai.

Le 31 décembre 1989, les âges de Jean et Claire sont les mêmes que dans la cas 1, et nous avons l'équation (1).

Le 1er juillet 1992, l'âge de Jean est $1991 - J$ et l'âge de Claire est $1992 - C$. Nous avons

$$1991 - J = 4(1992 - C). \quad (3)$$

Toujours en soustrayant (1) de (3), on a $2 = C - 1977$; c'est-à-dire, $C = 1979$ et $J = 1939$.

Ainsi, le 31 décembre 1989, Jean et Claire ont respectivement 50 et 10 ans, tandis qu'au 1er juillet 1992, ils ont 52 et 13 ans.

Il y a donc deux solutions :

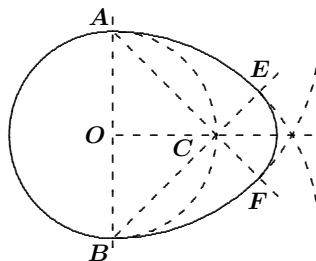
Naissance de Jean :	8 mai 1964,
Naissance de Claire :	21 août 1984,

ou

Naissance de Jean :	21 août 1939,
Naissance de Claire :	8 mai 1979.

Une solution erronée a été soumise.

7. (La poule géomètre.) Une figure plane en forme d'œuf est délimitée par quatre arcs de cercles désignés par \widehat{AB} , \widehat{BF} , \widehat{FE} et \widehat{EA} mis bout à bout de la façon indiquée par la figure à droite. Sachant que le rayon AO est de longueur 1, déterminer l'aire de la figure.



Solution officielle.

Notons que le segment $AC = \sqrt{2}$. L'aire de l'œuf A est égale à l'aire du demi-cercle ABO (de rayon 1) + l'aire du secteur BAF (rayon 2) + l'aire du secteur ABE (rayon 2) + l'aire du quart de cercle (rayon $(2 - \sqrt{2})$) - aire du triangle ABC (base 2 et hauteur 1). On obtient donc

$$A = \frac{1}{2}\pi + \frac{1}{8}\pi 2^2 + \frac{1}{8}\pi 2^2 + \frac{1}{4}\pi(2 - \sqrt{2})^2 - \frac{1}{2}2 = (3 - \sqrt{2})\pi - 1,$$

ce qui donne approximativement $A = 3,9819$.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Jeff Hooper (Acadia University). The Assistant Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are John Grant McLoughlin (University of New Brunswick), Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by 1 March 2008. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

We falsely attributed Mayhem proposal M276 [2007 : 8] to Babis Stergiou. The proposal was actually submitted by George Apostolopoulos, Mesologi, Greece, and we apologize to George for this oversight.

M304. Corrected. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a , b , and c be real numbers such that both $a + b + c$ and $ab + bc + ca$ are rational numbers, and $a + b + c \neq 0$. Show that $a^4 + b^4 + c^4$ is a rational number if and only if the product abc is a rational number.

M313. *Proposed by Babis Stergiou, Chalkida, Greece.*

Two circles with centres K and L intersect at points A and B . The tangent at A to the circle centred at L meets segment KB at M and the tangent at A to the circle centred at K meets segment BL at N . Prove that $AB \perp MN$.

M314. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a be a real number with $a > 1$. Solve the following equation for x :

$$a^{1/x}x + a^x/x = 2a.$$

M315. *Proposed by Mihály Bencze, Brasov, Romania.*

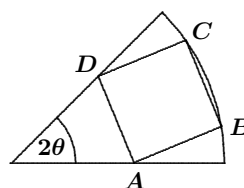
Let ABC be a triangle. Let D be the intersection of AB with the interior bisector of angle C , and let E be the mid-point of AB . Show that $CD + CE < AC + BC$.

M316. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the value of $\sum_{1 \leq k \leq 99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}}$.

M317. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Square $ABCD$ is inscribed in a sector of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc. The angle at the centre is 2θ . Determine the value of θ that results in the square of largest area.



M318. *Proposed by Houda Anoun, Bordeaux, France.*

Are there real numbers x and y such that $x^2 + xy = 3$ and $x - y^2 = 2$?

.....

M304. *Correction. Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a , b et c trois nombres réels tels que les sommes $a + b + c$ et $ab + bc + ca$ sont des nombres rationnels, et $a + b + c \neq 0$. Montrer qu'alors $a^4 + b^4 + c^4$ est un nombre rationnel si et seulement si le produit abc est un nombre rationnel.

M313. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Deux cercles de centres K et L se coupent aux points A et B . La tangente en A au cercle centré en L coupe le segment KB en M et la tangente en A au cercle centré en K coupe le segment BL en N . Montrer que $AB \perp MN$.

M314. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a un nombre réel avec $a > 1$. Résoudre l'équation suivante par rapport à x :

$$a^{1/x}x + a^x/x = 2a.$$

M315. *Proposé par Mihály Bencze, Brasov, Roumanie.*

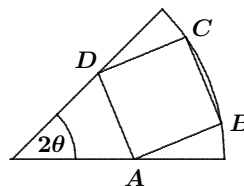
Dans un triangle ABC , soit D le point d'intersection de AB avec la bissectrice intérieure de l'angle C , et soit E le point milieu de AB . Montrer que $CD + CE < AC + BC$.

M316. *Proposé par Neven Jurič, Zagreb, Croatie.*

Trouver la valeur de $\sum_{1 \leq k \leq 99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}}$.

M317. *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

On inscrit un carré $ABCD$ dans un secteur d'un cercle de rayon 1 de sorte qu'il ait un sommet sur chacun des rayons frontières et deux sommets sur l'arc frontière. Si l'angle au centre vaut 2θ , déterminer la valeur de θ qui rend l'aire du carré maximale.



M318. *Proposé par Houda Anoun, Bordeaux, France.*

Existe-t-il des nombres réels x et y tels que $x^2 + xy = 3$ et $x - y^2 = 2$?

Mayhem Solutions

M263. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let a , b , and n be integers such that $(a^2 + b^2)/5 = n$. Prove that $n = c^2 + d^2$ for some integers c and d .

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, modified by the editor.

Without loss of generality, we assume that the remainder when a is divided by 5 is at least as large as the remainder when b is divided by 5. Since $5 \mid (a^2 + b^2)$, the possibilities for $(a, b) \pmod{5}$ are then $(0, 0)$, $(2, 1)$, $(3, 1)$, $(4, 2)$, and $(4, 3)$. We consider each of these cases in turn.

Case 1: $(a, b) \equiv (0, 0) \pmod{5}$.

Then $a = 5k$, $b = 5t$, where $k, t \in \mathbb{Z}$. Therefore,

$$n = \frac{a^2 + b^2}{5} = 5k^2 + 5t^2 = (2k + t)^2 + (k - 2t)^2,$$

from which we see that $n = c^2 + d^2$, where $c = 2k + t$ and $d = k - 2t$.

Case 2: $(a, b) \equiv (2, 1) \pmod{5}$.

Then $a = 5k + 2$, $b = 5t + 1$, where $k, t \in \mathbb{Z}$. Therefore,

$$n = \frac{a^2 + b^2}{5} = 5k^2 + 4k + 5t^2 + 2t + 1 = (2k + t + 1)^2 + (k - 2t)^2,$$

from which we see that $n = c^2 + d^2$, where $c = 2k + t + 1$ and $d = k - 2t$.

Case 3: $(a, b) \equiv (3, 1) \pmod{5}$.

Then $a = 5k + 3$, $b = 5t + 1$, where $k, t \in \mathbb{Z}$. Therefore,

$$n = \frac{a^2 + b^2}{5} = 5k^2 + 6k + 5t^2 + 2t + 2 = (2k - t + 1)^2 + (2t + k + 1)^2,$$

from which we see that $n = c^2 + d^2$, where $c = 2k - t + 1$ and $d = 2t + k + 1$.

Case 4: $(a, b) \equiv (4, 2) \pmod{5}$.

Then $a = 5k + 4$, $b = 5t + 2$, where $k, t \in \mathbb{Z}$. Therefore,

$$n = \frac{a^2 + b^2}{5} = 5k^2 + 8k + 5t^2 + 4t + 4 = (2k + t + 2)^2 + (2t - k)^2,$$

from which we see that $n = c^2 + d^2$, where $c = 2k + t + 2$ and $d = 2t - k$.

Case 5: $(a, b) \equiv (4, 3) \pmod{5}$.

Then $a = 5k + 4$, $b = 5t + 3$, where $k, t \in \mathbb{Z}$. Therefore,

$$n = \frac{a^2 + b^2}{5} = 5k^2 + 8k + 5t^2 + 6t + 5 = (k + 2t + 2)^2 + (2k - t + 1)^2,$$

from which we see that $n = c^2 + d^2$, where $c = k + 2t + 2$ and $d = 2k - t + 1$.

Therefore, if $(a^2 + b^2)/5 = n$, where $a, b, n \in \mathbb{Z}$, then $n = c^2 + d^2$ for some $c, d \in \mathbb{Z}$.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; and D. KIPP JOHNSON, Beaverton, OR, USA. One incomplete solution was also submitted.

Johnson's argument involved the fact that n can be written as the sum of two squares if and only if its prime factorization contains no odd powers of primes $q \equiv 3 \pmod{4}$. Since $5n = a^2 + b^2$, it must be possible to write n as a sum of two squares.

M264. Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.

Given 1001 real numbers placed around a circle such that each number is the arithmetic mean of its neighbours or else its two neighbours are equal, prove that all the numbers are equal.

Solution by D. Kipp Johnson, Beaverton, OR, USA, modified by the editor.

Let $x_1, x_2, \dots, x_{1001}$ denote 1001 real numbers placed around a circle in a fixed direction. For convenience, we regard the indices 1, 2, \dots , 1001 as integers modulo 1001 (in particular, $x_0 = x_{1001}$ and $x_{1002} = x_1$).

For each i , either $2x_i = x_{i-1} + x_{i+1}$ (if x_i is the arithmetic mean of its neighbours) or else $x_{i-1} = x_{i+1}$; in either case, $|x_{i+1} - x_i| = |x_i - x_{i-1}|$. Consequently, if we let $\delta = |x_2 - x_1|$, then we have $|x_{i+1} - x_i| = \delta$ for all i .

Note that $\sum_{i=1}^{1001} (x_{i+1} - x_i) = x_{1002} - x_1 = 0$, since $x_{1002} = x_1$. Each term of this sum is either δ or $-\delta$. If $\delta \neq 0$, then the number of positive terms in the sum must equal the number of negative terms (since the sum is 0). But this is impossible, because the sum has an odd number of terms. We conclude that $\delta = 0$.

Therefore, any two neighbouring numbers are equal, and thus, all of the numbers $x_1, x_2, \dots, x_{1001}$ must be equal.

There were two incorrect solutions submitted.

M265. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Given triangle ABC and $DE \parallel BC$, with $D \in AB$ and $E \in AC$. Drop perpendiculars from D and E to BC , meeting BC at F and K , respectively. If $\frac{[ABC]}{[DEKF]} = \frac{32}{7}$, determine the ratio $\frac{|AD|}{|DB|}$.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let AH be the altitude from A to BC , meeting DE at L . From $\frac{[ABC]}{[DEKF]} = \frac{32}{7}$, it follows that $\frac{1}{2} AH \cdot BC = \frac{32}{7} DE \cdot LH$; that is,

$$\frac{AH}{LH} = \frac{64}{7} \cdot \frac{DE}{BC}. \quad (1)$$

Since $DE \parallel BC$, triangles ABC and ADE are similar. Thus,

$$\frac{AD}{AB} = \frac{DE}{BC}. \quad (2)$$

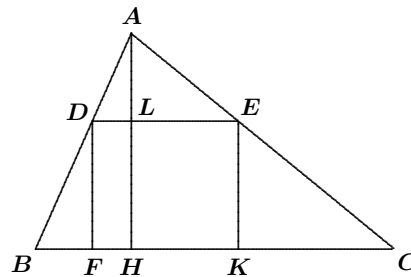
The condition $DE \parallel BC$ also implies that

$$\frac{AH}{LH} = \frac{AB}{DB}. \quad (3)$$

Substituting (2) and (3) into (1), we obtain $\frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{AB}$. Continuing to solve for $\frac{AB}{DB}$ leads to the following equivalent statements:

$$\begin{aligned} \frac{AB^2}{DB^2} &= \frac{64}{7} \cdot \frac{AD}{AB} \cdot \frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{DB}, \\ \left(\frac{AD+DB}{DB}\right)^2 &= \frac{64}{7} \cdot \frac{AD}{DB}, \\ \left(\frac{AD}{DB} + 1\right)^2 &= \frac{64}{7} \cdot \frac{AD}{DB}, \\ \left(\frac{AD}{DB}\right)^2 - \frac{50}{7} \frac{AD}{DB} + 1 &= 0, \end{aligned}$$

resulting in $\frac{AD}{DB} = \frac{1}{7}$ or $\frac{AD}{DB} = 7$.



Also solved by COURTIS G. CHRYSOSTOMOS, Larissa, Greece; and SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. There were three incomplete solutions submitted.

M266. Proposed by Bruce Sawyer, Memorial University of Newfoundland, St. John's, NL.

A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 17.
2. The sum of the two numbers is 89.
3. The product of the four digits is 210.
4. The product of the two numbers is 1924.

Determine the two numbers.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Since $1924 = 2^2 \cdot 13 \cdot 37$, there are exactly 12 positive divisors of 1924, namely 1, 2, 4, 13, 26, 37, 52, 74, 148, 481, 962, and 1924. Thus, the only factorizations of 1924 as the product of a pair of two-digit numbers are $26 \cdot 74$ and $37 \cdot 52$. It follows from any one of the first three properties that the numbers we seek are 37 and 52.

Remark: If we assume only properties 2 and 4, we need not make any assumption about the digits of the numbers: if two numbers x and y satisfy $x + y = 89$ and $xy = 1924$, it follows that the numbers are 37 and 52.

Also solved by COURTIS G. CHRYSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; D. KIPP JOHNSON, Beaverton, OR, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; VEDULA N. MURTY, Dover, PA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M267. Proposed by the Mayhem Staff.

Find a quintic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 7 repeated k times, then $f(n)$ consists of the digit 7 repeated $5k + 3$ times. (For example, $f(77) = 777777777777777$.) Compare with M256 [2006 : 265].

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Let $f(x)$ be a polynomial with the desired property. If n is a positive integer consisting of the digit 7 repeated k times, then $n = \frac{7}{9}(10^k - 1)$. We require $f(n)$ to consist of the digit 7 repeated $5k + 3$ times; that is, $f(n) = \frac{7}{9}(10^{5k+3} - 1)$.

Since $n = \frac{7}{9}(10^k - 1)$, we have $10^k - 1 = \frac{9}{7}n$, and thus $10^k = \frac{9}{7}n + 1$. Then

$$f(n) = \frac{7}{9}(1000 \cdot (10^k)^5 - 1) = \frac{7}{9} \left(1000 \cdot \left(\frac{9}{7}n + 1 \right)^5 - 1 \right).$$

Thus, $f(x) = \frac{7}{9} \left(1000 \cdot \left(\frac{9}{7}x + 1 \right)^5 - 1 \right)$.

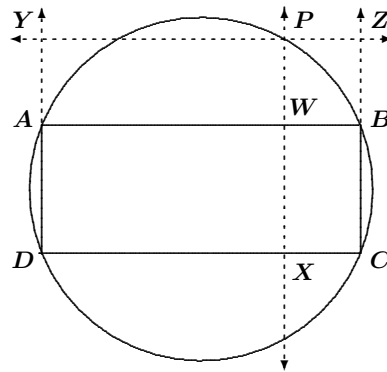
We can write $f(x)$ in a somewhat nicer form:

$$\begin{aligned} f(x) &= \frac{7000}{9} \left(\frac{9}{7}x + 1\right)^5 - \frac{7}{9} = \frac{7000}{9} \left(\left(\frac{9}{7}x + 1\right)^5 - 1\right) + \frac{7000}{9} - \frac{7}{9} \\ &= 777 + \frac{7000}{9} \left(\left(\frac{9}{7}x + 1\right)^5 - 1\right). \end{aligned}$$

Also solved by COURTIS G. CHRYSOSTOMOS, Larissa, Greece; and D. KIPP JOHNSON, Beaverton, OR, USA.

M268. Proposed by the Mayhem Staff.

Rectangle $ABCD$ is inscribed in a circle Γ and P is a point on Γ . Lines parallel to the sides of the rectangle are drawn through P and meet one pair of sides at points W and X and the extensions of the other pair of sides at Y and Z . Prove that the line through W and Y is perpendicular to the line through X and Z .



Similar solutions by Curtis G. Chryssostomos, Larissa, Greece; Hasan Denker, Istanbul, Turkey; and Vedula N. Murty, Dover, PA, USA.

If P is coincident with any of the four points A , B , C , or D , then the statement is true.

Suppose now that $P \notin \{A, B, C, D\}$. Let the centre of circle Γ be at $(0, 0)$, and let the coordinates of the vertices of the rectangle inscribed in Γ be $A(-a, b)$, $B(a, b)$, $C(a, -b)$, and $D(-a, -b)$, for some $a > 0$ and $b > 0$. Let P have coordinates (x_0, y_0) . The coordinates of points W , X , Y , and Z are then $W(x_0, b)$, $X(x_0, -b)$, $Y(-a, y_0)$, and $Z(a, y_0)$.

The equation of Γ is $x^2 + y^2 = a^2 + b^2$. Since $P(x_0, y_0)$ is on Γ , we have $x_0^2 + y_0^2 = a^2 + b^2$; that is,

$$\frac{x_0^2 - a^2}{y_0^2 - b^2} = -1. \quad (1)$$

The slope of the line through W and Y is $\frac{y_0 - b}{-a - x_0}$, and the slope of the line through X and Z is $\frac{y_0 + b}{a - x_0}$. The product of these two slopes is

$$\left(\frac{y_0 - b}{-a - x_0}\right) \cdot \left(\frac{y_0 + b}{a - x_0}\right) = \frac{y_0^2 - b^2}{x_0^2 - a^2} = -1,$$

where the last step uses (1). Thus, the lines WY and XZ are perpendicular.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; COURTIS G. CHRYSOSTOMOS, Larissa, Greece; and SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

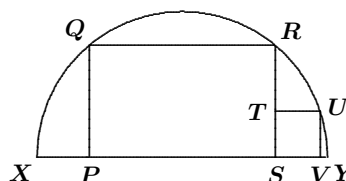
Problem of the Month

Ian VanderBurgh

Rectangles in circles or circles in rectangles? This month, we look at both and review some geometry.

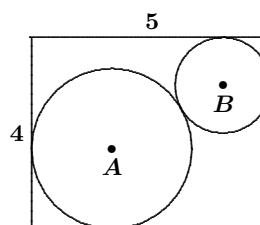
Problem 1 (2005 Fermat Contest)

In the diagram, a semicircle has diameter XY . Rectangle $PQRS$ is inscribed in the semicircle with $PQ = 12$ and $QR = 28$. Square $STUV$ has T on RS , U on the semicircle, and V on XY . What is the area of $STUV$?



Problem 2

In the diagram, the rectangle has height 4 and width 5, the circle with centre A has radius r , and the circle with centre B has radius 1. Each of the circles is tangent to two sides of the rectangle and to the other circle. Determine the value of r .



What are the important facts about circles that we need to know? What strategies should we use? These are good questions to ask (about whatever figures are in the problem) whenever attacking a problem in geometry.

There are really only three facts that we need to be able to attack either of these problems:

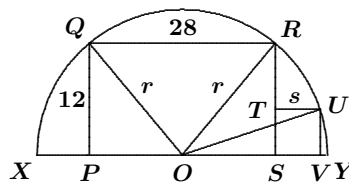
- (i) The distance from the centre of a circle to any point on its circumference equals the radius.
- (ii) If a circle is tangent to a line, the radius to the point of tangency forms a right angle with the line.
- (iii) If two circles are tangent, the straight line joining their centres passes through the point of tangency and so its length is the sum of the radii of the circles.

These are good facts, and relatively intuitive. But what strategies should we use? Since it would make sense to try to take advantage of these facts, three useful things to try are:

- (a) Joining the centre of a circle to useful points on the circumference.
- (b) Joining the centre of a circle to points of tangency with lines.
- (c) Joining the centres of mutually tangent circles.

As it turns out, the only mathematical machinery that we will need is the Pythagorean Theorem! This tends to be the case in these types of problems, even in problems that look a fair bit more complicated.

Solution to Problem 1: First, we mark the centre of the semi-circle, O , then we join O to Q , R and U . Let r be the radius of the semi-circle and s the side length of square $STUV$. Now, O appears to be half-way between P and S . Why is this true? Well, $PQ = SR$, $OQ = OR$ and $\angle QPO = \angle RSO = 90^\circ$; hence, $\triangle QPO$ and $\triangle RSO$ are congruent. Thus, $OP = OS = \frac{1}{2}PS = 14$.



By applying the Theorem of Pythagoras to $\triangle QPO$, we obtain $r^2 = 12^2 + 14^2 = 340$. (We could at this stage calculate r itself, but as it turns out, we won't need to.)

Next, we look at $\triangle OVU$. This triangle is right-angled at V with $OU = r$, $UV = s$, and $OV = OS + SV = 14 + s$. Therefore, by the Theorem of Pythagoras,

$$\begin{aligned} OU^2 &= UV^2 + OV^2, \\ r^2 &= s^2 + (s + 14)^2, \\ 340 &= s^2 + s^2 + 28s + 196, \\ 0 &= 2s^2 + 28s - 144, \\ 0 &= s^2 + 14s - 72, \\ 0 &= (s - 4)(s + 18); \end{aligned}$$

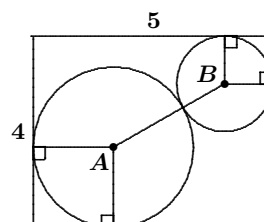
thus, $s = 4$ or $s = -18$. We reject the second solution, since s must be positive. Therefore, the area of square $STUV$ is 16.

For a complicated problem, we didn't have to use much machinery. Often, this is the case—a couple of judicious applications of the Pythagorean Theorem and the solution of a quadratic equation or two often does the trick.

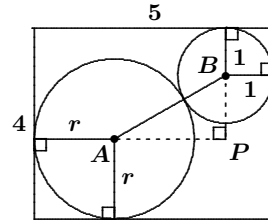
Of course, this problem was originally a multiple choice problem, and the proposers demonstrated a malicious streak by asking “The area of square $STUV$ is closest to . . .” and giving the choices 12, 13, 14, 15, and 16. Since the area was exactly 16 (instead of “close” to 16), one could be forgiven for being somewhat concerned! (At the same time, the “niceness” of the answer might lead one to believe that the answer was actually correct.)

Let's apply what we've learned to Problem 2.

Solution to Problem 2: Following our suggestions above, let's join A and B to points of tangency and to each other, and mark in right angles where we can. At this stage, it looks like we're stuck. But we had the hint above of trying to use the Pythagorean Theorem. To do that, we need a right-angled triangle. Do you see one? I don't, so let's build one, and mark in a few lengths too.



Where does this get us? First, we know that the length of AB is the sum of the radii of the circles, or $r + 1$. Now let's look at the height of the rectangle, which is 4. Breaking this into three pieces, it equals $r + PB + 1$; hence, $r + PB + 1 = 4$ or $PB = 3 - r$. (Extending BP to the bottom edge of the rectangle might help you see this.) Similarly, looking at the width, $r + AP + 1 = 5$, or $AP = 4 - r$.



Using the Pythagorean Theorem,

$$\begin{aligned} AB^2 &= AP^2 + PB^2, \\ (r + 1)^2 &= (4 - r)^2 + (3 - r)^2, \\ r^2 + 2r + 1 &= r^2 - 8r + 16 + r^2 - 6r + 9, \\ 0 &= r^2 - 16r + 24. \end{aligned}$$

Using the quadratic formula,

$$r = \frac{16 \pm \sqrt{16^2 - 4(1)(24)}}{2} = \frac{16 \pm \sqrt{160}}{2} = 8 \pm 2\sqrt{10}.$$

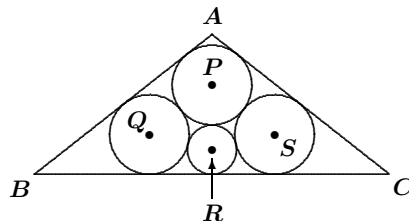
Since the circle with radius r is contained in the rectangle, the radius r cannot equal $8 + 2\sqrt{10}$ (which is larger than 14). Thus, $r = 8 - 2\sqrt{10}$.

So the same types of ideas worked in both problems, which is kind of nice. These techniques are relatively straightforward and do not require a lot of sophisticated mathematical knowledge. They do often require some insight (as in the construction of the right-angled triangle in the second problem), but that's where our problem-solving experience comes in most handy.

I leave you with a challenge problem, adapted from this year's Hypatia Contest. We'll look at the solution in next month's column. While this problem seems more complicated than the ones above, it can be solved using nothing more than the ideas we've seen so far.

Problem (2007 Hypatia Contest)

In the diagram, the circles with centres P , Q and S all have radius 1. Each is tangent to two sides of the isosceles $\triangle ABC$ and to the circle with centre R ; the circle with centre P is tangent to both of the other circles of radius 1. What is the radius of the circle with centre R ?



Pólya's Paragon

Now You See It, Now You Don't

Jeff Hooper

I'm sure you learned the knack of *cancelling* at some point. In fact, cancelling has probably become so second-nature that you do it quite without thinking about it. For instance, in a sum like $x^2 - 3x + 3x - 1$ or a fraction like $\frac{49}{35}$, eliminating the $3x$ terms from the sum or the factor 7 from the numerator and denominator of the fraction is almost automatic.

Cancelling can be of great benefit in solving problems, but sometimes it can hide some of the structure of a problem from us. No doubt you have already encountered one way this can happen, namely, the technique of completing the square.

For example, suppose we are asked to show that the expression $x^2 - x + \frac{1}{2}$ is positive for all real x . Completing the square *undoes* some simplifying to show that

$$x^2 - x + \frac{1}{2} = x^2 - 2\left(\frac{1}{2}x\right) + \frac{1}{4} + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}.$$

The right side can never be less than $1/4$, since the square is non-negative. In this case, inserting some additional terms allows us to rewrite the expression in a way which is more appropriate to the problem.

Problem 1. Show that for any positive real numbers x and y ,

$$4(x^3 + y^3) \geq (x + y)^3.$$

Solution 1: It's tempting to expand the right side here and work with the resulting expression:

$$4(x^3 + y^3) \geq x^3 + 3x^2y + 3xy^2 + y^3.$$

This can work, but again the resulting cancelling can sometimes eliminate too much.

We'll take another approach. The important idea here is that the expression $x^3 + y^3$ actually factors into $(x + y)(x^2 - xy + y^2)$. (Check that!) Using this factorization and cancelling a factor $(x + y)$ from each side (so we've temporarily assumed that $x \neq -y$), we can rewrite the inequality as

$$4(x^2 - xy + y^2) \geq (x + y)^2,$$

or, after simplifying,

$$3x^2 - 6xy + 3y^2 \geq 0.$$

Since this is equivalent to $3(x - y)^2 \geq 0$, we have reduced the inequality to something that must always hold. Thus, as in the first example, in a sense

we have *unsimplified* part of the expression in order to obtain our solution. [Strictly speaking of course, to finish up this problem, we need to show that these steps are all reversible, and take care of the assumption we introduced, but I will leave that for you.]

Sometimes there can be a great deal of *potential* cancelling in an expression, even if at first it is not obvious.

For example, consider the following sum:

$$\sum_{k=1}^{1000} \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

It is tempting to simplify the expression in brackets by combining the two terms, but that gets us nowhere quickly; although each expression reduces to a term which is not complicated, we still need to add 1000 such terms! In this case, it is far easier to write out the sum:

$$\left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{999} - \frac{1}{1000} \right) + \left(\frac{1}{1000} - \frac{1}{1001} \right).$$

Now you can see that the two $1/2$ s cancel, as do the two $1/3$ s, the two $1/4$ s, and so on, up to and including the two $1/1000$ s. The entire sum collapses leaving only the first and last terms. So we see that

$$\sum_{k=1}^{1000} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{1001} = \frac{1000}{1001}.$$

A sum such as this is called a *telescoping sum*, since this collapsing is a little like the way the sections of a small telescope collapse into one another. We will have a similar collapse with any sum that has the form

$$\sum_{k=1}^n (F(k) - F(k+1)) \quad \text{or} \quad \sum_{k=1}^n (F(k+1) - F(k)),$$

where $F(k)$ is some function of k . If we write this out in the longer form, the negative term in one bracket cancels with the positive term in the next.

The real power of this kind of cancellation shows itself when a more complicated expression can be rearranged into a telescoping form, as in the next example.

Problem 2. Compute the sum

$$\sum_{k=1}^{2007} \frac{1}{k(k+1)}.$$

This is again a large sum, but its terms certainly do not have the form $F(k) - F(k+1)$. Or do they? We need to put our simplification hat on backwards here (like we did earlier) and pull this term apart. We get

$$\frac{1}{k(k+1)} = \frac{(k+1) - k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Now, proceeding as in our example above, we get

$$\sum_{k=1}^{2007} \frac{1}{k(k+1)} = \sum_{k=1}^{2007} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{2008} = \frac{2007}{2008}.$$

A similar idea applies to products. The standard notation for products in mathematics is $\prod_{k=1}^n a_k$, which represents the product $a_1 a_2 a_3 \cdots a_n$. An expression like $\prod_{k=1}^{20} \frac{k+1}{k}$ can be written out as

$$\prod_{k=1}^{20} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{20}{19} \cdot \frac{21}{20}.$$

(Note that the *last* factor in the product has $k = 20$ in the denominator.) This time the cancelling is even easier to see, and we wind up with the answer 21.

We will have a similar collapse with any product that has the form

$$\prod_{k=1}^n \frac{F(k+1)}{F(k)} \quad \text{or} \quad \prod_{k=1}^n \frac{F(k)}{F(k+1)},$$

where, as before, $F(k)$ is some function of k . If we write out the product in the longer form, the numerator in one bracket cancels with the denominator in the next (or vice versa).

In dealing with such sums and products, the main difficulty is often rearranging everything into the correct form.

I'll close with a few problems for you to try yourself. (The last one will require at least one trigonometric identity.) You might even look through this month's Mayhem problems too!

1. Show that for any positive integer n , the value of $n^7 - n$ must always be a multiple of 7.
2. Find the sum

$$\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{2005} + \sqrt{2007}}.$$

3. Find the product

$$\prod_{k=1}^{100} \frac{k^2 + 4k + 4}{k^2 + 3k + 2}.$$

4. Simplify the expression

$$\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \frac{\tan 4}{\cos 8} + \cdots + \frac{\tan 128}{\cos 256}.$$

THE OLYMPIAD CORNER

No. 265

R.E. Woodrow

We begin this number of the *Corner* with the problems of the XXV Brazilian Mathematical Olympiad 2003, as translated by John Scholes (with minor edits). Thanks go to Christopher Small for collecting them for our use.

XXV BRAZILIAN MATHEMATICAL OLYMPIAD 2003

1. Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n .

2. Let S be a set with n elements. For a given positive integer k , for any distinct subsets A_1, A_2, \dots, A_k of S , and for each i , $1 \leq i \leq k$, choose $B_i = A_i$ or $B_i = S - A_i$. Find the smallest k such that we can always choose B_i so that $\bigcup_{1 \leq i \leq k} B_i = S$.

3. Let $ABCD$ be a rhombus. Let E, F, G , and H be points on the sides AB, BC, CD , and DA , respectively, so that EF and GH are tangent to the incircle of $ABCD$. Show that EH and FG are parallel.

4. Given a circle and a point A inside the circle, but not at its centre, find points B, C , and D on the circle which maximize the area of the quadrilateral $ABCD$.

5. Let $f(x)$ be a real-valued function defined on the positive reals such that

(i) $f(x) < f(y)$ if $x < y$, and

(ii) $f\left(\frac{2xy}{x+y}\right) = \frac{f(x) + f(y)}{2}$ for all x .

Show that $f(x) < 0$ for some value of x .

6. A graph G with n vertices is called *great* if we can label each vertex with a different positive integer not exceeding $\lfloor n^2/4 \rfloor$ and find a set of non-negative integers D so that there is an edge between two vertices if and only if the difference between their labels is in D . Show that if n is sufficiently large, we can always find a graph with n vertices which is not great.

Next, we give the problems of the first, second and third selection tests of the Republic of Moldova for IMO 2004. Thanks again go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

2004 REPUBLIC OF MOLDOVA First Selection Test

1. Suppose that the positive integer n has distinct representations as a sum of two squares of positive integers: $n = a^2 + b^2 = c^2 + d^2$. Prove that n is a composite number.

2. In a tetrahedron $ABCD$, let r be the radius of the inscribed sphere, and let r_A , r_B , r_C , and r_D be the radii of the spheres that are tangent to the faces of the tetrahedron and to the extensions of the other faces. Prove the inequality

$$\frac{1}{\sqrt{r_A^2 - r_A r_B + r_B^2}} + \frac{1}{\sqrt{r_B^2 - r_B r_C + r_C^2}} + \frac{1}{\sqrt{r_C^2 - r_C r_D + r_D^2}} + \frac{1}{\sqrt{r_D^2 - r_D r_A + r_A^2}} \leq \frac{2}{r}.$$

For what kind of tetrahedron does equality hold?

3. The circles Γ_1 and Γ_2 intersect each other at M and N . A straight line passing through M intersects the circle Γ_1 at $A \neq M$ and the circle Γ_2 at $B \neq M$, such that $M \in AB$. The internal bisector of the angle AMN meets the circle Γ_1 in the point D , and the internal bisector of the angle BMN meets the circle Γ_2 at C . Prove that the circle with CD as diameter passes through the mid-point of the segment AB .

4. Let n be a positive integer and $A = \{a_1, a_2, \dots, a_n\}$ be a set of real numbers. Find, in terms of n , the total number of functions $f : A \rightarrow A$ with the property $f(f(x)) - f(f(y)) \geq x - y$ for any $x, y \in A$ with $x > y$.

Second Selection Test

5. Let n be a positive integer, and let

$$A = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}_+^*, i = 1, 2, \dots, n\}.$$

A function $f : A \rightarrow \mathbb{R}$ is defined as follows: for all $(x_1, x_2, \dots, x_n) \in A$,

$$f(x_1, x_2, \dots, x_n) = \frac{1}{x_1} + \frac{1}{2x_2} + \frac{1}{3x_3} + \dots + \frac{1}{(n-1)x_{n-1}} + \frac{1}{nx_n}.$$

Show that $f(C_N^1, C_N^2, \dots, C_N^n) = f(2^{n-1}, 2^{n-2}, \dots, 2, 1)$, where C_n^k is the number of k -element subsets of an n -element set, for $k = 1, 2, \dots, n$.

6. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy the relation

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$$

for all real numbers x and y .

7. Let ABC be an acute-angled triangle with orthocentre H and circumcentre O . The inscribed and circumscribed circles have radii r and R , respectively. If P is an arbitrary point of the segment $[OH]$, prove that $6r \leq PA + PB + PC \leq 3R$.

8. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented in infinitely many ways as a sum of three distinct *good* integers whose product is the square of an odd integer.

Third Selection Test

9. For all positive real numbers a , b , and c , prove the inequality

$$\left| \frac{4(a^3 - b^3)}{a + b} + \frac{4(b^3 - c^3)}{b + c} + \frac{4(c^3 - a^3)}{c + a} \right| \leq (a - b)^2 + (b - c)^2 + (c - a)^2.$$

10. Determine all the polynomials $P(X)$ with real coefficients which satisfy the relation

$$(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)$$

for every real number x .

11. Let ABC be an isosceles triangle with $AC = BC$, and let I be its incentre. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC . The straight lines through P parallel to CA and CB meet AB at D and E , respectively. The line through P parallel to AB meets CA and CB at F and G , respectively. Prove that the straight lines DF and GE intersect on the circumcircle of the triangle ABC .

12. Let a_k be the number of integers n that satisfy the following conditions:

- (a) $n \in [0, 10^k)$; that is, n has exactly k digits (in decimal notation) with leading zeroes allowed;
- (b) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $a_{2m} = 10a_{2m-1}$ for every positive integer m .

Now we give the problems of the Sixth Hong Kong (China) Mathematical Olympiad, written December 20, 2003. Thanks again go to Christopher Small for collecting them for the *Corner*.

SIXTH HONG KONG (CHINA) MATHEMATICAL OLYMPIAD

December 20, 2003

Time: 3 hours

- 1.** Find the greatest real number K such that, for every positive u , v , and w with $u^2 > 4vw$,

$$(u^2 - 4vw)^2 > K(2v^2 - uv)(2w^2 - uv).$$

Justify your claim.

- 2.** Let $ABCDEF$ be a regular hexagon of side length 1, and let O be the centre of the hexagon. In addition to the sides of the hexagon, line segments are drawn from O to each vertex, making a total of twelve unit line segments. Find the number of paths of length 2003 along these line segments that start at O and terminate at O .

- 3.** Let $ABCD$ be a cyclic quadrilateral. Let K , L , M , and N be the mid-points of sides AB , BC , CD , and DA , respectively. Prove that the orthocentres of triangles AKN , BKL , CLM , and DMN are vertices of a parallelogram.

- 4.** Find, with reasons, all integers a , b , and c such that

$$\frac{1}{2}(a+b)(b+c)(c+a) + (a+b+c)^3 = 1 - abc.$$

As a fourth set of questions, we give the Final Round of the Swedish Mathematical Contest 2003/2004. Thanks again go to Christopher Small for obtaining them.

SWEDISH MATHEMATICAL CONTEST 2003-04

Final Round

November 22, 2003 Time: 5 hours No aids allowed.

- 1.** The numbers x , y , z , and w are all non-negative. Determine the smallest value of x such that the following relations hold:

$$y = x - 2003, \tag{1}$$

$$z = 2y - 2003, \tag{2}$$

$$w = 3z - 2003. \tag{3}$$

What are the corresponding values of y , z , and w ?

2. In a lecture hall, some chairs are placed in rows and columns, forming a rectangle. There are 6 boys in each row and there are 8 girls in each column, while 15 chairs are not occupied. What can you say about the number of rows and the number of columns?

3. Find all real numbers x which satisfy the equation

$$\lfloor x^2 - 2x \rfloor + 2\lfloor x \rfloor = \lfloor x \rfloor^2.$$

Here $\lfloor a \rfloor$ denotes the integer part of a (the largest integer not exceeding a).

4. Determine all polynomials P with real coefficients such that

$$1 + P(x) = \frac{1}{2}(P(x-1) + P(x+1))$$

for all real x .

5. Given two positive real numbers a and b , how many (non-congruent) plane quadrilaterals $ABCD$ are there such that $\angle B = 90^\circ$, $AB = a$, and $BC = CD = DA = b$?

6. Consider an infinite lattice of identical squares with an integer written in each square. Assume that, for each square, the integer within it is equal to the sum of the integer immediately above it and the integer immediately to the left of it. Assume also that there exists a row R_0 in the lattice such that *all numbers in R_0 are positive*. Denote by R_1 the row below R_0 , by R_2 the row below R_1 , etc. Show that, for each $N \geq 1$, the row R_N cannot contain more than N zeroes.

As a final set of problems to whet your problem-solving skills, we give the German Mathematical Olympiad, Final Round, Grades 12–13. Thanks once more to Christopher Small for collecting them.

2004 GERMAN MATHEMATICAL OLYMPIAD Final Round, Grades 12–13

1. Determine all pairs (x, y) of real numbers x and y which satisfy

$$\begin{aligned} x^4 + y^4 &= 17(x + y)^2, \\ xy &= 2(x + y). \end{aligned}$$

2. Let k be a circle with centre M . On k lies the point M_1 , which is the centre of another circle k_1 . Denote by g the line through M and M_1 . The point T lies on the circle k_1 in the interior of k . The tangent line t to k_1 at T intersects the circle k in the points A and B . Let a and b be the tangent lines to k_1 through A and B , respectively, which are different than t . Prove that either g , a , and b all intersect in a common point or they are parallel.

3. Prove that for any positive integer n , there is a positive integer z satisfying the following conditions:

- (i) The number z has exactly n digits.
- (ii) No digit of z equals 0.
- (iii) The number z is divisible by the sum of its digits.

4. For any positive integer n , let a_n denote the integer closest to \sqrt{n} . Determine

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{2004}}.$$

5. Prove that, for any positive real numbers a, b, c, d ,

$$a^3 + b^3 + c^3 + d^3 \geq a^2b + b^2c + c^2d + d^2a.$$

Determine when equality occurs.

6. A point (x, y) is called a lattice point if both x and y are integers. Determine (with proof) if there is a circle in the plane which contains exactly 5 lattice points.

Now we give a non-trigonometric solution for a problem from the Hungarian Mathematical Olympiad 2002-2003, discussed earlier this year [2006 : 150; 2007 : 161].

3. Let ABC be a triangle. We drop a perpendicular from A to the internal bisectors starting from B and C , their feet being A_1 and A_2 . In the same way we define B_1, B_2 and C_1, C_2 . Prove that

$$2(A_1A_2 + B_1B_2 + C_1C_2) = AB + BC + CA.$$

Alternate solution by J. Chris Fisher, University of Regina, Regina, SK.

We shall see that

$$A_1A_2 = s - a, \quad B_1B_2 = s - b, \quad C_1C_2 = s - c,$$

where s is the semiperimeter of $\triangle ABC$ and a, b, c are the sides. Then it will follow at once that

$$2(A_1A_2 + B_1B_2 + C_1C_2) = 2(s - a + s - b + s - c) = 2s,$$

which is the desired result.

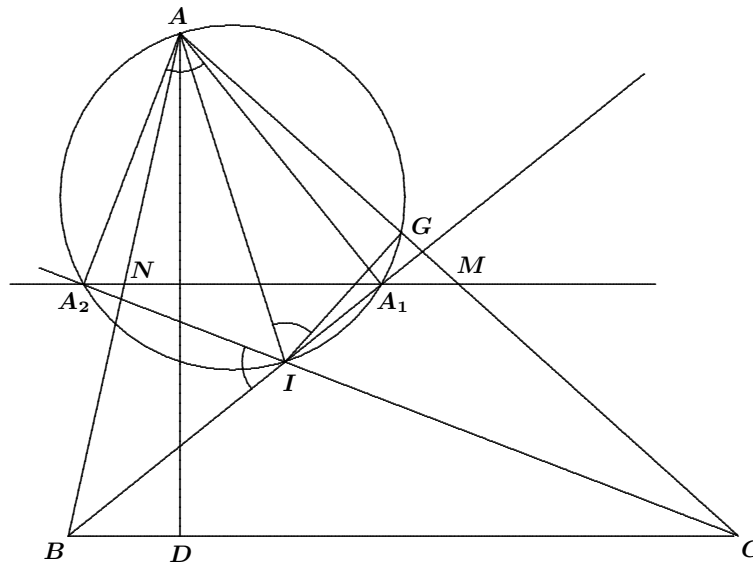
To this end, we denote the incentre (where BA_1 intersects CA_2) by I and look at the circle on diameter AI . Because of the right angles at A_1 and A_2 , the quadrangle AA_2IA_1 is cyclic; whence, $\angle A_2AA_1 = \angle A_2IB$. This last angle is an exterior angle of $\triangle BIC$, so that

$$\angle A_2AA_1 = \angle A_2IB = \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} - \frac{A}{2}.$$

Let G be the foot of the perpendicular from I to AC . Then $\angle GIA$ is the complement of $\angle IAG$ in the right triangle IAG , which implies that

$$\angle A_2AA_1 = \angle GIA.$$

Because G is the point where the incircle of $\triangle ABC$ touches the side AC , we have $AG = s - a$. Furthermore, AI subtends the right angle at G , so that G is another point on the circle AA_2IA_1G whose diameter is AI . Because the inscribed angles $\angle A_2AA_1$ and $\angle GIA$ are equal, the chords that subtend them, namely A_1A_2 and AG must have the same length; that is, $A_1A_2 = s - a$ as claimed. Similarly $B_1B_2 = s - b$ and $C_1C_2 = s - c$, which completes the proof.



Comment. Compare this problem with the result discussed by Bruce Shawyer in his *Mayhem* article "Remarkable Bisections" [2006 : 434–435]. Shawyer proved that the line A_1A_2 is the perpendicular bisector of the altitude AD . As a consequence A_1A_2 is the line joining the mid-points M and N of the sides AB and AC . An easy way to see this is to note that the circle whose diameter is AC (with centre M and radius MA) passes through A_2 and D ; the bisector of $\angle ACD$ meets the perpendicular bisector of AD (namely MN) at the point A_2 of the circle.

Now we turn to solutions from our readers to problems given in the November 2006 number of the *Corner* and the problems of Category A of the Belarus Mathematical Olympiad 2002, Final Round, given [2006 : 436–437].

2. (*D. Bazylev*) Let

$$P(x) = (x+1)^p(x-3)^q = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where p and q are positive integers.

- (a) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.
 (b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $P(x)$.

Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Ioannis Katsikis, Athens, Greece. We present the solution of Bornshtein.

(a) The roots of $P(x)$ are evidently -1 and 3 , with respective multiplicities p and q (where $p + q = n$). Denote these roots by r_1, r_2, \dots, r_n , where $r_1 = r_2 = \cdots = r_p = -1$ and $r_{p+1} = r_{p+2} = \cdots = r_n = 3$.

Using the well-known relations between roots and coefficients, we have

$$\sum_{i=1}^n r_i = -a_1 \quad \text{and} \quad \sum_{1 \leq i < j \leq n} r_i r_j = a_2.$$

On the other hand, using the known values of the roots, we obtain $\sum_{i=1}^n r_i = -p + 3q$ and

$$\begin{aligned} \sum_{1 \leq i < j \leq n} r_i r_j &= \binom{p}{2}(-1)^2 + pq(-1)(3) + \binom{q}{2}(3)^2 \\ &= \frac{1}{2}p(p-1) - 3pq + \frac{9}{2}q(q-1). \end{aligned}$$

Thus,

$$a_1 = p - 3q \quad \text{and} \quad a_2 = \frac{1}{2}p(p-1) - 3pq + \frac{9}{2}q(q-1).$$

Therefore, $a_1 = a_2$ if and only if

$$p^2 - 3p - 6pq + 9q^2 - 3q = 0. \quad (1)$$

From (1), we deduce that p is divisible by 3, which in turn forces q to be a multiple of 3. Let $p = 3a$ and $q = 3b$.

Thus, $a_1 = a_2$ if and only if $a^2 - a - 6ab + 9b^2 - b = 0$, which is equivalent to $(a - 3b)^2 = a + b$. It follows that $a_1 = a_2$ if and only if $3n = 9(a + b) = 9(a - 3b)^2$, and we are done.

(b) From above, $a_1 = a_2$ if and only if $(a - 3b)^2 = a + b$, where $p = 3a$ and $q = 3b$. Let $a - 3b = x$. Then $a_1 = a_2$ if and only if

$$(a, b) = \left(\frac{3}{4}x(x+1), \frac{1}{4}x(x-1) \right). \quad (2)$$

Since a and b are positive integers, we choose any positive integer t and, letting $x = 4t$ in (2), we deduce that, for $(p, q) = (36t^2 + 3t, 12t^2 - 3t)$, the polynomial $P(x)$ satisfies $a_1 = a_2$.

4. (V. Kolbun) Positive numbers a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n satisfy the condition $a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n = 1$.

Find the smallest possible value of the sum

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \dots + \frac{a_n^2}{a_n + b_n}.$$

Solved by Michel Bataille, Rouen, France; and Pierre Bornshtein, Maisons-Laffitte, France. We give Bataille's version.

Let S be the given sum. We prove that the smallest value of S is $\frac{1}{2}$. Since $S = \frac{1}{2}$ for $a_i = b_i = \frac{1}{n}$ ($i = 1, 2, \dots, n$), the proof will be complete if we show that $S \geq \frac{1}{2}$.

Let

$$T = \frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2} + \dots + \frac{a_n b_n}{a_n + b_n}.$$

Note that, for $i = 1, 2, \dots, n$,

$$\frac{a_i b_i}{a_i + b_i} \leq \frac{a_i + b_i}{4},$$

since $(a_i + b_i)^2 - 4a_i b_i = (a_i - b_i)^2 \geq 0$. Therefore,

$$T \leq \frac{1}{4} \sum_{i=1}^n (a_i + b_i) = \frac{1}{2}.$$

Now

$$S = \sum_{i=1}^n \left(a_i - \frac{a_i b_i}{a_i + b_i} \right) = 1 - T \geq \frac{1}{2}.$$

6. (A. Romanenko, D. Zmeikov)

(a) A positive integer is called *nice* if it can be represented as an arithmetic mean of some (not necessarily distinct) positive integers each of which is a non-negative power of 2.

Prove that all positive integers are nice.

(b) A positive integer is called *ugly* if it cannot be represented as an arithmetic mean of pairwise *distinct* positive integers each of which is a non-negative power of 2.

Prove that there exist infinitely many ugly positive integers.

Solution by Pierre Bornshtein, Maisons-Laffitte, France.

(a) For each positive integer k , let \mathcal{N}_k be the set of all the nice positive integers which can be represented as an arithmetic mean of 2^k positive integers each of which is a non-negative power of 2. Let $\mathcal{N} = \bigcup_{k \geq 1} \mathcal{N}_k$. We will prove that $\mathcal{N} = \mathbb{N}^*$.

Lemma 1. For each $k \geq 1$, we have $\mathcal{N}_k \subseteq \mathcal{N}_{k+1}$.

Proof: This follows from the equality

$$\frac{\sum 2^i}{2^k} = \frac{\sum 2^i + \sum 2^i}{2^{k+1}},$$

where each of the sums has exactly 2^k summands. ■

Lemma 2. If $x, y \in \mathcal{N}$ then $x + y \in \mathcal{N}$.

Proof: Using Lemma 1, we may find $k \geq 1$ such that $x, y \in \mathcal{N}_k$. Let

$$x = \frac{\sum 2^i}{2^k} \quad \text{and} \quad y = \frac{\sum 2^j}{2^k},$$

where each of the sums has exactly 2^k summands. Then

$$x+y = \frac{\sum 2^i + \sum 2^j}{2^k} = \frac{\sum 2^{i+1} + \sum 2^{j+1}}{2^{k+1}} \in \mathcal{N}_{k+1} \subseteq \mathcal{N}. \quad \blacksquare$$

We have $1 = \frac{2^0}{1} \in \mathcal{N}$. Using Lemma 2, we deduce that if $x \in \mathcal{N}$, then $x + 1 \in \mathcal{N}$. Then, by induction, $\mathcal{N} = \mathbb{N}^*$.

(b) Let \mathcal{U} be the set of all ugly positive integers.

Lemma 3. For any positive integer n , we have $n \in \mathcal{U}$ if and only if $2n \in \mathcal{U}$.

Proof: We will prove that $n \notin \mathcal{U}$ if and only if $2n \notin \mathcal{U}$.

First, suppose $n \notin \mathcal{U}$. Then $n = \frac{1}{k} \sum_{i \in I} 2^i$ for some set $I \subseteq \mathbb{N}$ with $|I| = k$. Thus,

$$2n = \frac{1}{k} \sum_{i \in I} 2^{i+1},$$

so that $2n \notin \mathcal{U}$.

Now suppose $2n \notin \mathcal{U}$. Then $2n = \frac{1}{k} \sum_{i \in I} 2^i$ for some set $I \subseteq \mathbb{N}$ with $|I| = k$. Since the exponents are distinct, it follows that at most one is 0; hence, at most one of the numbers 2^i which appear in the sum is odd. But the whole sum has to be even. Thus, none of the exponents is 0. Therefore, each of the exponents is at least 1. Then $n = \frac{1}{k} \sum_{i \in I} 2^{i-1}$, and $n \notin \mathcal{U}$. ■

It follows from Lemma 3 that we only have to find one ugly number, say x , because for such a number we deduce that $2^p x$ is ugly for each non-negative integer p .

We now show 13 is ugly.

Assume, for a contradiction, that $13 \notin \mathcal{U}$. Then there exist $k \in \mathbb{N}^*$ and $I \subseteq \mathbb{N}$ with $|I| = k$ such that

$$13k = \sum_{i \in I} 2^i. \quad (1)$$

It follows that $13k \geq 1 + 2 + \dots + 2^{k-1} = 2^k - 1$. But, a straightforward induction shows that $2^n - 1 > 13n$ for all integers $n \geq 7$. This gives $k \leq 6$.

Since the exponents in (1) are distinct, equation (1) is the binary expansion of $13k$. One can now verify that, for each $k = \{1, \dots, 6\}$, the binary expansion of $13k$ does not have exactly k non-zero digits, which leads to the desired contradiction.

7. (*E. Barabanov*) Does there exist a surjective function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the expression $f(x+y) - f(x) - f(y)$ takes exactly two values 0 and 1 for various real x and y ?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Yes. For example, let

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 0. \end{cases}$$

Then f is clearly surjective. Now take any $x \leq y$.

(i) If $x \leq y \leq 0$, then $f(x+y) = x+y = f(x) + f(y)$.

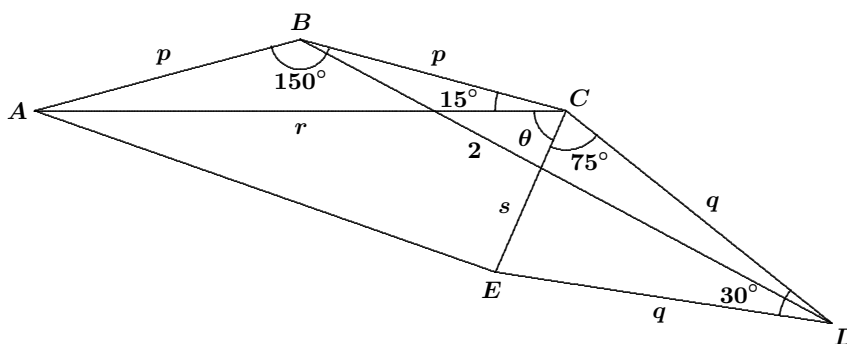
(ii) If $x \leq 0 < y$ and $x+y \leq 0$, then $f(x+y) = x+y = f(x) + f(y) + 1$.

(iii) If $x \leq 0 < y$ and $x+y > 0$, then $f(x+y) = x+y-1 = f(x) + f(y)$.

(iv) If $0 < x \leq y$, then $f(x+y) = x+y-1 = f(x) + f(y) + 1$.

8. (*I. Voronovich*) Find the area of the convex pentagon $ABCDE$, given that $AB = BC$, $CD = DE$, $\angle ABC = 150^\circ$, $\angle CDE = 30^\circ$, and $BD = 2$.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give Kandall's write-up.



Let $p = AB = BC$ and $q = CD = DE$. Let $r = AC$, $s = CE$, and $\theta = \angle ACE$. From the given information, we see that $\angle ACB = 15^\circ$ and $\angle DCE = 75^\circ$. Then $r = 2p \cos 15^\circ$ and $s = 2q \cos 75^\circ = 2q \sin 15^\circ$. Thus, $rs = 2pq \cdot 2 \sin 15^\circ \cos 15^\circ = 2pq \sin 30^\circ = pq$. By applying the Cosine Law to $\triangle BCD$, we get

$$4 = p^2 + q^2 - 2pq \cos(\theta + 90^\circ) = p^2 + q^2 + 2pq \sin \theta.$$

Consequently,

$$\begin{aligned}
 [ABCDE] &= [ABC] + [CDE] + [ACE] \\
 &= \frac{1}{2}p^2 \sin 150^\circ + \frac{1}{2}q^2 \sin 30^\circ + \frac{1}{2}rs \sin \theta \\
 &= \frac{1}{4}(p^2 + q^2 + 2pq \sin \theta) = \frac{1}{4}(4) = 1.
 \end{aligned}$$

Now we turn to some solutions to problems from Category B, Belarus Mathematical Olympiad 2002, Final Round given at [2006 : 437–438].

4. (*I. Voronovich*) Pairwise distinct positive integers a, b, c, d, e, f, g, h , and n satisfy the equalities $n = ab + cd = ef + gh$.

Find the smallest possible value of n .

Solved by Ioannis Katsikis, Athens, Greece.

The smallest possible value of n is 47.

The idea is to use numbers as small as possible; that is, the numbers 1, 2, 3, 4, 5, 6, 7, trying to find numbers which can be expressed twice in the form of a product of two factors.

We have $12 = 2 \cdot 6 = 3 \cdot 4$ and $35 = 5 \cdot 7 = 1 \cdot 35$

Thus, if we take $a = 2, b = 6, c = 5, d = 7, e = 3, f = 4, g = 1$, and $h = 35$, then, for $n = 47$, we have $n = ab + cd = ef + gh$.

5. (*I. Voronovich*) The quadrilateral $ABCD$ is cyclic and has the property that $AB = BC = AD + CD$. Given that $\angle BAD = \alpha$ and that the diagonal $AC = d$, find the area of the triangle ABC .

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution by Bataille, modified by the editor.

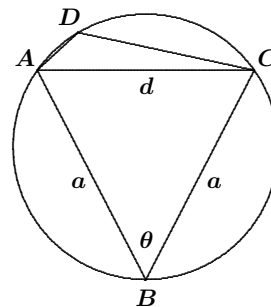
We will show that $[ABC] = \frac{1}{2}d^2 \sin \alpha$ (here and in what follows, $[\cdot]$ denotes area).

Let $a = AB = BC = AD + CD$ and $\theta = \angle ABC$. Then

$$\begin{aligned}
 [ABCD] &= [DAB] + [DCB] \\
 &= \frac{1}{2}AB \cdot AD \sin \alpha + \frac{1}{2}BC \cdot CD \sin \alpha \\
 &= \frac{1}{2}a(AD + CD) \sin \alpha = \frac{1}{2}a^2 \sin \alpha
 \end{aligned}$$

and

$$\begin{aligned}
 [ABCD] &= [ABC] + [ADC] = \frac{1}{2}AB \cdot BC \sin \theta + \frac{1}{2}AD \cdot CD \sin \theta \\
 &= \frac{1}{2}(a^2 + AD \cdot CD) \sin \theta
 \end{aligned}$$



Thus,

$$a^2 \sin \alpha = (a^2 + AD \cdot CD) \sin \theta. \quad (1)$$

By the Law of Cosines,

$$d^2 = AB^2 + BC^2 - 2AB \cdot BC \cos \theta = 2a^2(1 - \cos \theta);$$

that is, $2(1 - \cos \theta) = d^2/a^2$. Also,

$$\begin{aligned} d^2 &= AD^2 + CD^2 + 2AD \cdot CD \cos \theta \\ &= (AD + CD)^2 - 2AD \cdot CD(1 - \cos \theta) = a^2 - AD \cdot CD(d^2/a^2), \end{aligned}$$

which implies that $a^2 + AD \cdot CD = a^4/d^2$. Using this result in (1), we get $a^2 \sin \alpha = (a^4/d^2) \sin \theta$; that is, $d^2 \sin \alpha = a^2 \sin \theta$.

$$\text{Finally, } [ABC] = \frac{1}{2}AB \cdot BC \sin \theta = \frac{1}{2}a^2 \sin \theta = \frac{1}{2}d^2 \sin \alpha.$$

[Ed.: What is the answer if the quadrilateral is non-convex?]

Next we look at solutions from our readers to problems of Category C, Belarus Mathematical Olympiad 2002, Final Round, given at [2006 : 439].

1. (A. Mirotin) (a) There are $k \geq 3$ positive integers such that no two of them are coprime while any three of them are coprime. Determine all possible values of k .

(b) Does there exist an infinite set of positive integers satisfying the same condition?

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

(a) This is possible for each $k \geq 3$.

Let $p_1, p_2, \dots, p_n, \dots$ be the sequence of primes, in increasing order. Let $k \geq 3$ be an integer.

Write in a row all the pairs (i, j) for $1 \leq i < j \leq k$, and label them $1, 2, \dots, \frac{1}{2}k(k-1)$ from left to right. For $i \in \{1, \dots, k\}$, let x_i be the product of all the primes p_n for which n is the label of a pair containing i .

If $i, j \in \{1, \dots, k\}$ with $i < j$, then x_i and x_j have the common factor p_r , where r is the label of the pair (i, j) ; thus, x_i and x_j are not coprime. On the other hand, p_r divides none of the other numbers x_1, \dots, x_k ; thus, any three of these numbers are coprime.

(b) No, there is no infinite set satisfying the condition.

Assume, for a contradiction, that $\{x_1, x_2, \dots\}$ is such an infinite set. The number x_1 has a finite number of prime divisors. For at least one of these divisors, say d , we must have $\gcd(x_1, x_j) = d$ for infinitely many j (because x_1 is not coprime with any of the other x_j s). Without loss of generality, we assume that $d = \gcd(x_1, x_2) = \gcd(x_1, x_3)$. It follows that $\gcd(x_1, x_2, x_3) = d > 1$, which contradicts the condition that any three of the x_i s are coprime.

2. (*I. Zhuk*) Prove that a right-angled triangle can be inscribed in the parabola $y = x^2$ so that its hypotenuse is parallel to the x -axis if and only if the altitude from the right angle is equal to 1. (A triangle is inscribed in a parabola if all three vertices of the triangle are on the parabola.)

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Michel Bataille, Rouen, France. We give Bataille's solution, modified by the editor.

Let ABC be a triangle with $\angle A = 90^\circ$, placed so that the hypotenuse BC is parallel to the x -axis and the vertices B and C lie on the parabola $y = x^2$. Suppose that the coordinates of B are (b, b^2) , where $b > 0$. Then the coordinates of C are $(-b, b^2)$. Let (x_0, y_0) be the coordinates of A .

Note that $\overrightarrow{AB} = [b - x_0, b^2 - y_0]$ and $\overrightarrow{AC} = [-b - x_0, b^2 - y_0]$. Since $\angle A = 90^\circ$, we have $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$; that is,

$$x_0^2 - b^2 + (b^2 - y_0)^2 = 0. \quad (1)$$

If $\triangle ABC$ is inscribed in the parabola $y = x^2$, then A lies on the parabola, which implies that $y_0 = x_0^2$. This, together with (1), gives $y_0 - b^2 + (b^2 - y_0)^2 = 0$, or $(b^2 - y_0)(b^2 - y_0 - 1) = 0$. Since $b^2 \neq y_0$, we obtain $b^2 - y_0 = 1$, which means that the altitude from A is equal to 1.

Conversely, if the altitude from A is equal to 1, then $b^2 - y_0 = 1$. Setting $b^2 = y_0 + 1$ in (1), we obtain $x_0^2 - (y_0 + 1) + 1 = 0$, or $y_0 = x_0^2$. Thus, A lies on the parabola and $\triangle ABC$ is inscribed in the parabola.

3. (*I. Zhuk*) The diagonals A_1A_4 , A_2A_5 , and A_3A_6 of the convex hexagon $A_1A_2A_3A_4A_5A_6$ meet at a point K . Given that $A_2A_1 = A_2A_3 = A_2K$, $A_4A_3 = A_4A_5 = A_4K$, and $A_6A_5 = A_6A_1 = A_6K$, prove that the hexagon is cyclic.

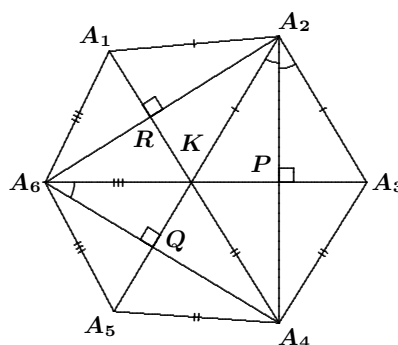
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We first observe that A_2A_4 is the perpendicular bisector of KA_3 , A_4A_6 is the perpendicular bisector of KA_5 , and A_6A_2 is the perpendicular bisector of KA_1 . Let the points at which these perpendicular bisections occur be denoted by P , Q , and R , respectively. Then

$$\triangle A_2PA_3 \cong \triangle A_2PK \sim \triangle A_6QK.$$

Thus,

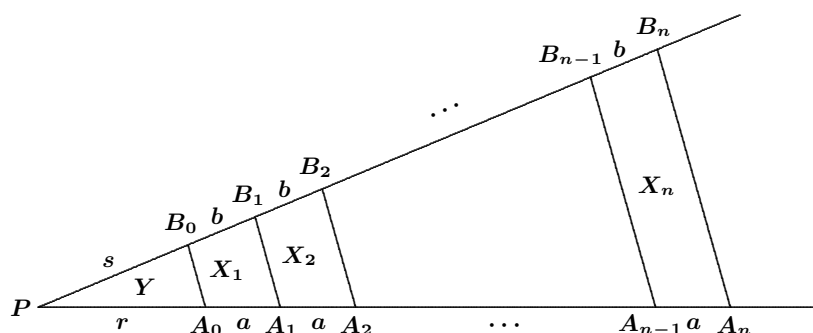
$$\angle A_3A_6A_4 = \angle A_5A_2A_4 = \angle A_3A_2A_4.$$



Hence, the quadrilateral $A_2A_3A_4A_6$ is cyclic; that is, A_3 lies on the circumcircle of $\triangle A_2A_4A_6$. Similarly, A_5 and A_1 lie on the same circumcircle. Therefore, the hexagon is cyclic.

6. (A. Shamruk) Distinct points $A_0, A_1, \dots, A_{1000}$ on one side of an angle and distinct points $B_0, B_1, \dots, B_{1000}$ on the other side are spaced so that $A_0A_1 = A_1A_2 = \dots = A_{999}A_{1000}$ and $B_0B_1 = B_1B_2 = \dots = B_{999}B_{1000}$. Find the area of the quadrilateral $A_{999}A_{1000}B_{1000}B_{999}$ if the areas of the quadrilaterals $A_0A_1B_1B_0$ and $A_1A_2B_2B_1$ are equal to 5 and 7, respectively.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let P denote the vertex of the angle, and let $r = PA_0$, $a = A_0A_1$, $s = PB_0$, $b = B_0B_1$, $Y = [PA_0B_0]$, and $X_n = [A_{n-1}A_nB_nB_{n-1}]$, where $[\cdot]$ denotes area, as usual. Furthermore, set $\alpha = a/r$ and $\beta = b/s$. We have

$$\begin{aligned} \frac{X_n}{Y} &= \frac{[PA_nB_n]}{Y} - \frac{[PA_{n-1}B_{n-1}]}{Y} \\ &= \frac{(r+na)(s+nb)}{rs} - \frac{(r+(n-1)a)(s+(n-1)b)}{rs} \\ &= (1+n\alpha)(1+n\beta) - (1+(n-1)\alpha)(1+(n-1)\beta) \\ &= \alpha + \beta + (2n-1)\alpha\beta; \end{aligned}$$

that is, $X_n = (\alpha + \beta + (2n-1)\alpha\beta)Y$. Consequently, $X_{n+1} - X_n = 2\alpha\beta Y$. Thus, X_1, X_2, X_3, \dots is an arithmetic progression.

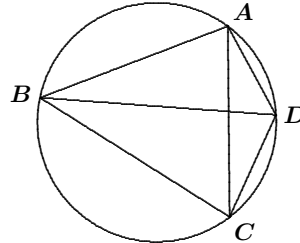
In the case where $X_1 = 5$ and $X_2 = 7$, the common difference is 2; therefore, $X_{1000} = 5 + 999 \cdot 2 = 2003$.

7. (I. Voronovich) A quadrilateral $ABCD$ is cyclic with $AB = 2AD$ and $BC = 2CD$. Given that $\angle BAD = \alpha$, and diagonal $AC = d$, find the area of the triangle ABC .

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Ioannis Katsikis, Athens, Greece. We give the solution of Bataille.

Note that A and C lie on the circle consisting of all points P such that $PB = 2PD$, which is centred on the line BD . It follows that A and C are on opposite sides of BD (otherwise, we would have $A = C$); that is, $ABCD$ is convex.

Denote area by $[\cdot]$. We will show that $[ABC] = \frac{1}{2}d^2 \sin \alpha$.



First,

$$[ABC] = \frac{1}{2}BA \cdot BC \sin \angle ABC = 2AD \cdot CD \sin \angle ADC = 4[ACD];$$

thus,

$$[ABC] = \frac{4}{5}[ABCD]. \quad (1)$$

Now,

$$\begin{aligned} [ABCD] &= [ABD] + [DCB] = \frac{1}{2}AB \cdot AD \sin \alpha + \frac{1}{2}CB \cdot CD \sin \alpha \\ &= (AD^2 + CD^2) \sin \alpha. \end{aligned} \quad (2)$$

From the Law of Cosines, we have

$$d^2 = AD^2 + CD^2 - 2AD \cdot CD \cos \angle ADC$$

and also

$$\begin{aligned} d^2 &= AB^2 + BC^2 - 2AB \cdot BC \cos \angle ABC \\ &= 4AD^2 + 4CD^2 + 8DA \cdot DC \cos \angle ADC, \end{aligned}$$

from which it readily follows that $8(AD^2 + CD^2) = 5d^2$. With (1) and (2), this immediately yields

$$[ABC] = \frac{4}{5} \cdot \frac{5}{8}d^2 \cdot \sin \alpha = \frac{1}{2}d^2 \sin \alpha.$$

That completes the *Corner* for this time. Please send your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Mathematical Delights

By Ross Honsberger, Mathematical Association of America, 2004
ISBN 088385-334-5, softcover, x + 252 pages, US\$41.00.

Reviewed by **Ed Barbeau**, University of Toronto, Toronto, ON

There was a time when Ross Honsberger of the University of Waterloo performed a mathematical concert at each annual meeting of the Ontario Association for Mathematics Education. Eager mathematics teachers would pack a large auditorium for a polished and witty exposition of about ten of Honsberger's favourite problems and their solutions, selected for their elegance and capacity to surprise and delight. Those who show up at the annual marking bee for the Waterloo contests still can enjoy such a treat.

These problems found their way into a succession of books published by the Mathematical Association of America. No fewer than eleven of the first twenty-eight volumes of the Dolciani Mathematical Exposition Series, including the inaugural four and this one, are from his hand. That is a lot of beautiful mathematics!

While his earlier books consisted of longer essays on individual problems, this one is a miscellaneous collection of problems from a variety of sources, briefly treated. Demanding at most the background of a second-year undergraduate, the author aims to "put on display little gems that are to be found at the elementary level". The first part of the book, *Gleanings*, contains problems and solutions drawn from contests like the Putnam, journals like *Mathematics Magazine* and *The College Mathematics Journal*, and published collections of problems. The second part, *Miscellaneous Topics*, focuses on the work of particular correspondents (Liong-shin Hahn, Achilleas Sinefakopoulos and George Evagelopoulos) and problems from particular sources (New Mexico Mathematics Contest of 2002, and *The Book of Prime Number Records* by Paulo Ribenboim). Finally, just to make sure the reader is not content to be a spectator, Honsberger poses 27 challenges, with solutions provided in a separate section.

As you would expect, the problems are drawn from the standard competition areas of number theory, combinatorics, algebra, and geometry. They are attractive for different reasons. Sometimes the result itself surprises. (As Honsberger often asked in his lectures, "How does someone think of such things?") At other times, there is an unusual strategy leading to a straightforward *dénouement*. But the most satisfying solutions are clever, unexpected, and brief. Sometimes a serious research problem has such a solution. Witness this question of M.V. Subbarao of the University of Alberta: *Are there $r \geq 2$ distinct odd primes p_1, p_2, \dots, p_r and an integer a for which $(p_1 + a)(p_2 + a) \cdots (p_r + a) - 1$ is divisible by $(p_1 + a - 1)(p_2 + a - 1) \cdots (p_r + a - 1)$?* A \$100 award went to C. Offord

and R. Wentz for an almost trivial example where $r = 2$ and the primes are twins.

In part, the book celebrates the human ingenuity that generated the problems and solutions, the latter occasionally during a competition. For example, the 1988 IMO problem to show that $(a^2 + b^2)/(ab + 1)$ is square whenever a and b are integers for which $ab + 1$ divides $a^2 + b^2$ was a notoriously challenging one for which a Bulgarian student gave a prize-winning solution during the competition.

The geometry problems are the most fun. There are a number of intriguing results about the sizes of circles inside an arbelos (a region bounded by three tangent semicircles with a common diameter). From *The College Mathematics Journal* come two short constructions for the tangent to an ellipse from an exterior point.

The book has an index for names and another for terms, with each item keyed to the section rather than the page containing it.

aha! A two volume collection

By Martin Gardner, Mathematical Association of America, 2006

ISBN 13: 978-0-88385-551-5, hardbound, 368 pages, US\$47.50

Reviewed by **Amar Sodhi**, Sir Wilfred Grenfell College, Corner Brook, NL

As a teenager, I would eagerly look forward to reading the new Martin Gardner book that came to the public library or bookstore. Each chapter was taken from Gardner's insightful column in *Scientific American*. I would joyfully spend an hour or so to read and digest the material contained therein.

In the *aha!* series, however, paradoxes (in part 1) and puzzles (in part 2) are presented in a series of vignettes. Each vignette is accompanied by a cartoon strip which introduces the reader to the problem being discussed.

The topics touched on in this work will no doubt be familiar to the older readers of *CRUX with MAYHEM*, but this does not matter. Gardner's inimitable style ensures that the knowledgeable reader can enjoy the book as if it were a collection of much loved poems. Even a reader who has little exposure to *aha!* aspects of mathematics may, like my wife and teenage daughter, find this entertaining yet thought-provoking book hard to put down.

Originally published in separate volumes as: *aha! Insight* (W.H. Freeman and Company, 1978) and *aha! Gotcha* (W.H. Freeman and Company, 1982), this welcome amalgamation of these classics makes for an ideal gift for anyone, young or old, who has yet to discover Martin Gardner.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 May 2008**. An asterisk (*) after a number indicates that a problem was proposed without a solution.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3276. Proposed by Neven Jurič, Zagreb, Croatia.

A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers satisfies the recurrence relation $a_{n+3} = a_{n+1} + a_n$ for $n \geq 0$. Simplify

$$\sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}.$$

3277. Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

The Lucas numbers L_n satisfy the recurrence relation $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Let k be an even positive integer. Find

$$\lim_{n \rightarrow \infty} \left(\{ \sqrt[k]{L_n} \} - \{ \sqrt[k]{L_{n-k}} + \sqrt[k]{L_{n-2k}} \} \right),$$

where $\{x\}$ is the fractional part of x (that is, $\{x\} = x - [x]$, where $[x]$ is the integer part of x).

3278. Proposed by Virgil Nicula, Bucharest, Romania.

Let P be a point in the plane of $\triangle ABC$ such that $PC = PB$ and $PA = AB$. Let x be the measure of $\angle PBC$. Prove that

$$\sin(B - C) = 2 \sin C \cos(B + 2\epsilon x),$$

where $\epsilon = 1$ if the line BC separates the points P and A , and $\epsilon = -1$ otherwise.

3279. Proposed by Virgil Nicula, Bucharest, Romania.

Let O , I , R , and r be the circumcentre, incentre, circumradius, and inradius of $\triangle ABC$, and let a , b , and c be the lengths of the sides of $\triangle ABC$ opposite the angles A , B , and C , respectively. Let IO meet the lines AB and AC at M and N , respectively. Prove that the points B , C , M , and N are concyclic if and only if $h_a = R + r$ (where h_a is the altitude to the side BC), and, in this case, we also have $\frac{1}{MN} = \frac{1}{a} + \frac{1}{b+c}$.

3280. Proposed by Virgil Nicula, Bucharest, Romania.

Let O and R be the circumcentre and circumradius, respectively, of $\triangle ABC$. Let E and F be points on AB and AC , respectively, such that O is the mid-point of segment EF . Let A' be the point where the line AO meets the circumcircle Γ of $\triangle ABC$ a second time, and let P be the point on the line EF such that $A'P \perp EF$. Prove that the lines EF , BC , and the tangent line to Γ at A' are concurrent, and that $\angle BPA' = \angle CPA'$.

3281. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.

Let a_1, a_2, \dots, a_n be positive real numbers. Prove that

$$\left(\sum_{k=1}^n a_k^{\frac{n+1}{2}} \right)^n \leq \prod_{k=1}^n \left(\sum_{k=1}^n a_k^k \right).$$

3282. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.

Let $A(z)$ be a polynomial of degree n with complex coefficients. Suppose the zeroes z_1, z_2, \dots, z_n of $A(z)$ are distinct non-zero complex numbers. Prove that

$$\sum_{k=1}^n \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{z_k - z_j} = 0.$$

3283. Proposed by M.N. Deshpande, Nagpur, India.

Of the $n!$ permutations σ of $(1, 2, \dots, n)$, for how many is σ^3 the identity permutation?

3284. Proposed by K.S. Bhanu and M.N. Deshpande, Institute of Sciences, Nagpur, India.

Let x, y , and z be positive real numbers which satisfy $x^2 + y^2 = z^2$. Construct a line segment AC with length z . Let B be any point such that $BC = x$ and $90^\circ < \angle ABC < 180^\circ$. Let M be a point on AC such that $\angle MAB = \angle MBC$. Let D be the point on line BM on the opposite side of AC from B such that $AD = y$. Show that $\angle ADM = \angle DCM$.

3285. Proposed by Gregory Akulov, student, University of Regina, Regina, SK.

Solve the following for x :

$$x \left(\sqrt{3 - 2x + \sqrt{5(1 - x^2)}} + \sqrt{\frac{3}{2}} \right) = \sqrt{\frac{2}{3}}.$$

3286. *Proposed by Neven Jurič, Zagreb, Croatia.*

Is it possible to find a function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = 1 + x \int_0^1 f(t) dt + x^2 \int_0^1 [f(t)]^2 dt?$$

3287. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $x, y,$ and z be positive real numbers satisfying

$$xy + yz + zx + xyz = 4.$$

Prove that

(a) $(x+2)(y+2) + (y+2)(z+2) + (z+2)(x+2) = (x+2)(y+2)(z+2);$

(b) there is a triangle whose sides have lengths $(x+2)(y+2), (y+2)(z+2),$ and $(z+2)(x+2).$

3288. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let n be a positive integer. Evaluate the sum:

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \frac{2^{n-2i-1}}{n-2i},$$

where $\lfloor x \rfloor$ is the integer part of x .

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3276. *Proposé par Neven Jurič, Zagreb, Croatie.*

Sachant que la suite $\{a_n\}_{n=0}^\infty$ de nombres réels positifs obéit à la relation de récurrence $a_{n+3} = a_{n+1} + a_n$ pour $n \geq 0$, simplifier

$$\sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2}.$$

3277. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Les nombres de Lucas L_n satisfont la relation de récurrence $L_0 = 2, L_1 = 1,$ et $L_{n+2} = L_{n+1} + L_n$ pour $n \geq 0$. Soit k un entier positif pair. Trouver

$$\lim_{n \rightarrow \infty} \left(\left\{ \sqrt[k]{L_n} \right\} - \left\{ \sqrt[k]{L_{n-k}} + \sqrt[k]{L_{n-2k}} \right\} \right),$$

où $\{x\}$ est la partie fractionnaire de x (c'est-à-dire, $\{x\} = x - \lfloor x \rfloor$, où $\lfloor x \rfloor$ est la partie entière de x).

3278. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit P un point dans le plan du triangle ABC tel que $PC = PB$ et $PA = AB$. Soit x la mesure de l'angle PBC . Montrer que

$$\sin(B - C) = 2 \sin C \cos(B + 2\varepsilon x),$$

où $\varepsilon = 1$ si la droite BC sépare les points P et A , et $\varepsilon = -1$ sinon.

3279. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit respectivement O, I, R et r les centres et rayons des cercles circonscrits et inscrits du triangle ABC , et a, b et c les longueurs des côtés du triangle ABC opposés aux angles A, B et C . Désignons par M et N les points où IO coupe les droites AB et AC . Montrer que les points B, C, M et N sont sur un même cercle si et seulement si $h_a = R + r$ (où h_a est la hauteur abaissée sur le côté BC) et que, dans ce cas, on a aussi $\frac{1}{MN} = \frac{1}{a} + \frac{1}{b+c}$.

3280. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit respectivement O et R le centre et le rayon du cercle circonscrit au triangle ABC . Soit E sur AB et F sur AC deux points tels que O soit le milieu du segment EF . Soit A' le point où AO coupe le cercle circonscrit Γ du triangle ABC une deuxième fois, et soit P le point sur la droite EF tel que $A'P \perp EF$. Montrer que les droites EF, BC et la tangente à Γ en A' sont concourantes, et que $\angle BPA' = \angle CPA'$.

3281. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit a_1, a_2, \dots, a_n des nombres réels positifs. Montrer que

$$\left(\sum_{k=1}^n a_k^{\frac{n+1}{2}} \right)^n \leq \prod_{k=1}^n \left(\sum_{k=1}^n a_k^k \right).$$

3282. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit $A(z)$ un polynôme de degré n à coefficients complexes. Supposons que les zéros z_1, z_2, \dots, z_n de $A(z)$ sont des nombres complexes non nuls distincts. Montrer que

$$\sum_{k=1}^n \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{z_k - z_j} = 0.$$

3283. *Proposé par M.N. Deshpande, Nagpur, Inde.*

Des $n!$ permutations σ de $(1, 2, \dots, n)$, combien y en a-t-il de sorte que σ^3 soit la permutation identité ?

3284. *Proposé par K.S. Bhanu et M.N. Deshpande, Institut des Sciences, Nagpur, Inde.*

Soit x , y et z trois nombres réels positifs satisfaisant $x^2 + y^2 = z^2$. On dessine un segment AC de longueur z . Soit B un point tel que $BC = x$ et $90^\circ < \angle ABC < 180^\circ$. Soit M un point sur AC tel que $\angle MAB = \angle MBC$. Soit finalement D le point sur la droite BM du côté opposé à AC par rapport à B de sorte que $AD = y$. Montrer que $\angle ADM = \angle DCM$.

3285. *Proposé par Gregory Akulov, étudiant, Université de Regina, Regina, SK.*

Trouver pour quel x :

$$x \left(\sqrt{3 - 2x + \sqrt{5(1 - x^2)}} + \sqrt{\frac{3}{2}} \right) = \sqrt{\frac{2}{3}}.$$

3286. *Proposé par Neven Jurič, Zagreb, Croatie.*

Est-il possible de trouver une fonction $f : [0, 1] \rightarrow \mathbb{R}$ telle que

$$f(x) = 1 + x \int_0^1 f(t) dt + x^2 \int_0^1 [f(t)]^2 dt ?$$

3287. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit x , y et z trois nombres réels positifs satisfaisant

$$xy + yz + zx + xyz = 4.$$

Montrer que

- (a) $(x+2)(y+2) + (y+2)(z+2) + (z+2)(x+2) = (x+2)(y+2)(z+2)$;
 (b) il existe un triangle dont les côtés ont comme longueur $(x+2)(y+2)$, $(y+2)(z+2)$ et $(z+2)(x+2)$.

3288. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Soit n un entier positif. Evaluer la somme :

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \frac{2^{n-2i-1}}{n-2i},$$

où $\lfloor x \rfloor$ est la partie entière de x .

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3027. [2005 : 173, 176; 2006 : 185–186] *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $ABCD$ be any quadrilateral, and let M be the mid-point of AB . On the sides CB , DC , and AD , equilateral triangles CBE , DCF , and ADG are constructed externally. Let N be the mid-point of EF and P be the mid-point of FG .

Prove that $\triangle MNP$ is equilateral.

Ed: In the comments following the solution previously featured for this problem [2006 : 185–186], we challenged our readers to find a purely geometric solution to the problem. We now have such a solution.

Solution by Waldemar Pompe, University of Warsaw, Poland.

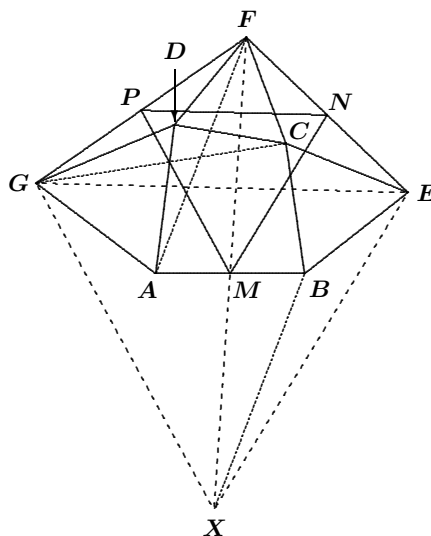
Let X be a point such that triangle EGX is equilateral, as shown in the diagram. Consider the rotation R_D with centre D and angle -60° . This rotation takes the points A and F to G and C , respectively. Next consider the rotation R_E with centre E and angle 60° . It takes the points G and C to X and B , respectively.

Thus, the composition $R_E \circ R_D$ is a translation which takes the points A and F to X and B , respectively. Therefore, $AFBX$ is a parallelogram, which implies that the point M , the mid-point of AB , is also the mid-point of FX .

Now consider the homothety with centre F and scale factor $\frac{1}{2}$. It takes the equilateral triangle EGX to the triangle NPM . Hence, $\triangle NPM$ is also equilateral.

Remark: One can similarly prove the following generalization:

Let $ABCD$ be any quadrilateral, and let M be the mid-point of AB . On the sides CB , DC , and AD , similar triangles CBE , CFD , and GAD are constructed externally, such that $\angle BCE = \angle FCD = \angle AGD = \alpha$ and $\angle CEB = \angle CDF = \angle GDA = \beta$. Let N and P be the mid-points of EF and FG , respectively. Then $\angle MPN = \alpha$ and $\angle PNM = \beta$.



3176. [2006 : 462, 464] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A_1A_2 \dots A_{4n}$ be a planar polygon with perimeter $4n$, where n is a positive integer. Prove that this polygon can be covered by a circle with radius n .

Solution by Mohammed Aassila, Strasbourg, France.

Let $ABCD$ be any minimal rectangle covering the polygon, and let r and s be the lengths of the sides AB and BC . There must be a vertex of the polygon on each side of the rectangle, for otherwise, the rectangle would not be minimal (it is possible that a vertex of the polygon coincides with a vertex of the rectangle, say B ; in this case the vertex of the polygon is on both sides AB and BC). Let the four vertices of the polygon on the sides of the rectangle be $X \in AB$, $Y \in BC$, $Z \in CD$, and $U \in DA$. As per our earlier remark, a pair or two of these vertices might coincide with a vertex or a pair of opposite vertices of the rectangle, respectively.

Let x , y , z , and u be the lengths of the segments AX , BY , CZ , and DU , respectively, and let $P(A_1A_2 \dots A_{4n})$ be the perimeter of the polygon. Using Minkowski's Inequality, we have

$$\begin{aligned} 4n &= P(A_1A_2 \dots A_{4n}) \geq P(XYZU) \\ &= \sqrt{x^2 + (s-u)^2} + \sqrt{u^2 + (r-z)^2} \\ &\quad + \sqrt{z^2 + (s-y)^2} + \sqrt{y^2 + (r-x)^2} \\ &\geq \sqrt{[x + (r-x) + z + (r-z)]^2 + [(s-u) + u + (s-y) + y]^2} \\ &= 2\sqrt{r^2 + s^2} = 2AC, \end{aligned}$$

which shows that our rectangle has a diagonal $AC \leq 2n$, and therefore, it can be covered by a circle with radius n . Consequently, the polygon can be covered by a circle of radius n .

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous remarked that this problem is a special case of the more general result that a planar figure of circumference C has a diameter less than $\frac{1}{2}C$, but he did not give a reference. The editor was able to locate a slightly modified version of the current problem as #97b in [1].

References

- [1] D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *Geometricheskie otzhenki i zadachi iz kombinatornoi geometrii*, Moskva, Nauka, 1975, pp. 313-314.

3178. [2006 : 462, 464] *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all integers x , y , z such that $4^x + 4^y + 4^z$ is a perfect square.

Comment by Michel Bataille, Rouen, France.

This was problem E2075 in the *American Mathematical Monthly*. Its solution appeared in Vol. 76, No. 3, March 1969, p. 308.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, OSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; and the proposer. There was one incomplete solution.

Janous proposed the more general problem of determining all integers x , y , and z such that $a^x + a^y + a^z$ is a perfect square, where a is an integer and $a > 1$. He notes that there are infinitely many bases a for which solutions exist. He provides two examples: $x = 0$, $y = z = 1$ (the expression $2a + 1$ is a perfect square for infinitely many a), and $x = 0$, $y = z = 2$ (the expression $2a^2 + 1$ is a perfect square for infinitely many a , from the Pell Equation $b^2 - 2a^2 = 1$).

3179. [2006 : 462, 464] Proposed by Michel Bataille, Rouen, France.

A transversal of $\triangle ABC$ makes angles α , β , and γ with the lines BC , CA , and AB , respectively. Express the minimum and maximum values of

$$(\cos \alpha \cos \beta \cos \gamma)^2 + (\sin \alpha \sin \beta \sin \gamma)^2$$

as functions of $p = \cos A \cos B \cos C$.

Solution by the proposer.

Let $X = (\cos \alpha \cos \beta \cos \gamma)^2 + (\sin \alpha \sin \beta \sin \gamma)^2$, and let \vec{U} be a non-zero vector on the transversal. Let A , B , and C be the oriented angles $\angle(\vec{AB}, \vec{AC})$, $\angle(\vec{BC}, \vec{BA})$, and $\angle(\vec{CA}, \vec{CB})$, respectively, and let x , y , and z be the oriented angles $\angle(\vec{BC}, \vec{U})$, $\angle(\vec{CA}, \vec{U})$, and $\angle(\vec{AB}, \vec{U})$, respectively. Note that, modulo 2π , the value of x is in the set $\{\alpha, -\alpha, \pi - \alpha, \alpha - \pi\}$, so that $\cos^2 \alpha = \cos^2 x$ and $\sin^2 \alpha = \sin^2 x$. Now, again modulo 2π , we have

$$y = \angle(\vec{CA}, \vec{BC}) + \angle(\vec{BC}, \vec{U}) = x + C + \pi$$

and

$$z = \angle(\vec{AB}, \vec{BC}) + \angle(\vec{BC}, \vec{U}) = x + \pi - B.$$

Thus,

$$\begin{aligned} X &= \cos^2 x \cos^2(x + C) \cos^2(x - B) + \sin^2 x \sin^2(x + C) \sin^2(x - B) \\ &= \frac{1}{8}(1 + \cos 2x)[\cos(2x + C - B) - \cos A]^2 \\ &\quad + \frac{1}{8}(1 - \cos 2x)[\cos(2x + C - B) + \cos A]^2 \\ &= \frac{1}{4} \cos^2(2x + C - B) + \frac{1}{4} \cos^2 A \\ &\quad - \frac{1}{2} \cos 2x \cos(2x + C - B) \cos A \\ &= \frac{1}{8}[1 + \cos(4x + 2C - 2B)] + \frac{1}{4} \cos^2 A \\ &\quad - \frac{1}{4} \cos A[\cos(4x + C - B) + \cos(C - B)] \\ &= \frac{1}{8} \cos(4x + 2C - 2B) - \frac{1}{4} \cos A \cos(4x + C - B) \\ &\quad + \frac{1}{8} + \frac{1}{4} \cos A[\cos A - \cos(C - B)]. \end{aligned}$$

Using the easy-to-prove relation $\cos A - \cos(C - B) = -2 \cos B \cos C$, we obtain

$$\begin{aligned} 8X &= \cos(4x + 2C - 2B) - 2 \cos A \cos(4x + C - B) + 1 - 4p \\ &= M \cos 4x + N \sin 4x + 1 - 4p, \end{aligned}$$

where

$$\begin{aligned} M &= \cos(2C - 2B) - 2 \cos A \cos(C - B) \\ \text{and } N &= 2 \cos A \sin(C - B) - \sin(2C - 2B). \end{aligned}$$

We easily find that $M^2 + N^2 = 1 - 8p$. Then

$$8X = \sqrt{1 - 8p} \cos(4x + \varphi) + 1 - 4p,$$

for some φ . It follows immediately that the minimum and maximum values of X are $\frac{1}{8}(1 - 4p - \sqrt{1 - 8p})$ and $\frac{1}{8}(1 - 4p + \sqrt{1 - 8p})$, respectively.

Note: If $\triangle ABC$ is equilateral, then $p = \frac{1}{8}$, and therefore, the value of X is independent of the transversal and constantly equal to $\frac{1}{16}$. This particular case was the object of a problem posed by V. Thébault in *Journal de mathématiques élémentaires*, 1950–1951, p. 75.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3180. [2006 : 462, 464] Proposed by Michel Bataille, Rouen, France.

Find all positive real numbers a such that

$$(\sqrt{a} + 3)^{\frac{1}{5}} + (\sqrt{5} - 2)^{\frac{1}{3}} = (\sqrt{a} - 3)^{\frac{1}{5}} + (\sqrt{5} + 2)^{\frac{1}{3}}.$$

Composite of similar solutions by Brian D. Beasley, Presbyterian College, Clinton, SC, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Set $\alpha = \frac{\sqrt{5} + 1}{2}$ and $\beta = \frac{\sqrt{5} - 1}{2}$. Then $\alpha - \beta = \alpha\beta = 1$. Since $\alpha^3 = \sqrt{5} + 2$ and $\beta^3 = \sqrt{5} - 2$, we have $(\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}} = 1$. Let $x = (\sqrt{a} - 3)^{\frac{1}{5}}$. Then the given equation becomes

$$(x^5 + 6)^{\frac{1}{5}} = x + 1. \quad (1)$$

Since $x^5 + 6 = \sqrt{a} + 3 > 3$, we have $x + 1 > 1$; whence, $x > 0$.

Equation (1) is equivalent to $x^5 + 6 = (x + 1)^5$. Expanding and simplifying gives $x^4 + 2x^3 + 2x^2 + x - 1 = 0$; that is,

$$(x^2 + x + \alpha)(x^2 + x - \beta) = 0.$$

Clearly, $x^2 + x + \alpha$ has no real roots, since $1 - 4\alpha < 0$. The only positive root of $x^2 + x - \beta$ is

$$x = \frac{-1 + \sqrt{1 + 4\beta}}{2} = \frac{-1 + \sqrt{2\sqrt{5} - 1}}{2},$$

which leads to the only solution:

$$a = (x^5 + 3)^2 = \left[\left(\frac{-1 + \sqrt{2\sqrt{5} - 1}}{2} \right)^5 + 3 \right]^2 = \frac{7 + 5\sqrt{5}}{2}.$$

[Ed: The last step can be verified either by brute force computations or by using a computer algebra system.]

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was also one incorrect solution.

Demis and Hess gave the answer as $a = 9 + \left(\frac{1+\sqrt{5}}{2}\right)^{-5}$, while Howard and Johnson gave $a = 9 + \left(\frac{-1+\sqrt{5}}{2}\right)^5$.

3181. [2006 : 462, 464] Proposed by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.

Show that for every integer $n \geq 2$, the equation $x^n + x^{-n} = 1 + x$ has a root in the interval $\left(1, 1 + \frac{1}{n}\right)$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC, USA.

Let $f(x) = x^n + x^{-n} - x - 1$. First we show that $f\left(1 + \frac{1}{n}\right) > 0$. For $n = 2$, we have $f\left(1 + \frac{1}{2}\right) = \frac{7}{36}$. For $n \geq 3$, the Binomial Theorem implies that

$$\left(1 + \frac{1}{n}\right)^n > 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = 2 + \frac{n-1}{2n} \geq 2 + \frac{1}{n};$$

thus, $f\left(1 + \frac{1}{n}\right) > \left(1 + \frac{1}{n}\right)^n - \left(1 + \frac{1}{n}\right) - 1 > 0$.

Next, we let $g(x) = x^n f(x) = x^{2n} - x^{n+1} - x^n + 1$. Then it is easy to verify that $g(x) = (x-1)h(x)$, where

$$h(x) = x^{2n-1} + x^{2n-2} + \dots + x^{n+2} + x^{n+1} - x^{n-1} - x^{n-2} - \dots - x - 1.$$

Since $f\left(1 + \frac{1}{n}\right) > 0$, we deduce that $g\left(1 + \frac{1}{n}\right) > 0$ and $h\left(1 + \frac{1}{n}\right) > 0$. But $h(1) = n - 1 - n = -1 < 0$; thus, the Intermediate Value Theorem implies that there exists $r \in \left(1, 1 + \frac{1}{n}\right)$ such that $h(r) = 0$. It then follows that $f(r) = 0$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; "THE THIRD FLOOR", Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most solvers used the facts that $f(1) = 0$ and $f'(1) = -1$ to deduce that $f(1 + \epsilon) < 0$ for sufficiently small $\epsilon > 0$.

Lau showed that f has only one root exceeding one, and Woo established the stronger result that in fact, $r \in \left(1, 1 + \frac{1}{n(n-1)}\right)$.

3182. Replacement. [2007 : 40, 43] Proposed by Arkady Alt, San Jose, CA, USA.

Let a , b , and c be any positive real numbers, and let p be a real number such that $0 < p < 1$.

(a) Prove that

$$\frac{a}{(b+c)^p} + \frac{b}{(c+a)^p} + \frac{c}{(a+b)^p} \geq \frac{1}{2^p} (a^{1-p} + b^{1-p} + c^{1-p}).$$

(b) Prove that, if $p = 1/3$, then

$$\frac{a}{(a+b)^p} + \frac{b}{(b+c)^p} + \frac{c}{(c+a)^p} \geq \frac{1}{2^p} (a^{1-p} + b^{1-p} + c^{1-p}).$$

(c)★ Prove or disprove

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{1}{\sqrt{2}} (\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Solution to part (a) by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since the proposed inequality is homogeneous, we may suppose that $a + b + c = 1$. This yields the equivalent inequality

$$\frac{a}{(1-a)^p} + \frac{b}{(1-b)^p} + \frac{c}{(1-c)^p} \geq \frac{1}{2^p} (a^{1-p} + b^{1-p} + c^{1-p}).$$

Without loss of generality, we may assume that $a \geq b \geq c$. It then follows that

$$\frac{1}{1-a} \geq \frac{1}{1-b} \geq \frac{1}{1-c}.$$

Using Chebyshev's Inequality and the AM–GM Inequality, we have

$$\begin{aligned} & \frac{a}{(1-a)^p} + \frac{b}{(1-b)^p} + \frac{c}{(1-c)^p} \\ & \geq \frac{a+b+c}{3} \left[\frac{1}{(1-a)^p} + \frac{1}{(1-b)^p} + \frac{1}{(1-c)^p} \right] \\ & \geq \frac{1}{((1-a)(1-b)(1-c))^{p/3}} \geq \frac{1}{\left[\frac{1-a+1-b+1-c}{3} \right]^p} = \frac{3^p}{2^p}. \end{aligned}$$

Thus, we need only prove that $a^{1-p} + b^{1-p} + c^{1-p} \leq 3^p$.

Let $f(x) = x^{1-p}$. Since $f''(x) = -p(1-p)x^{-p-1}$ and $0 < p < 1$, we see that $f''(x) < 0$ for $0 < x < 1$. Hence, f is a concave function on $(0, 1)$. Using Jensen's Inequality, we get

$$a^{1-p} + b^{1-p} + c^{1-p} \leq 3f\left(\frac{1}{3}(a+b+c)\right) = 3f\left(\frac{1}{3}\right) = 3^p.$$

[Ed.: Equality holds if and only if $a = b = c$.]

Solution to part (b) by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

We will first prove that, for all $x > 0$, the following inequality holds

$$\frac{4x^3 \sqrt[3]{2}}{\sqrt[3]{x^3+1}} \geq 5x^2 - 1. \quad (1)$$

This inequality is trivial if $x \leq 1/\sqrt{5}$. For $x > 1/\sqrt{5}$, define

$$f(x) = \frac{128x^9}{(x^3+1)(5x^2-1)^3}.$$

Proving inequality (1) is equivalent to showing that $f(x) \geq 1$. We compute

$$f'(x) = \frac{384x^8(1-x)(2x^2-3x-3)}{(x^3+1)^2(5x^2-1)^4},$$

from which we see that $f'(x) > 0$ for $1 < x < \frac{1}{4}(3 + \sqrt{33})$ and $f'(x) < 0$ for $1/\sqrt{5} < x < 1$ and for $x > \frac{1}{4}(3 + \sqrt{33})$. Thus,

$$f(x) \geq \min \left\{ f(1), \lim_{x \rightarrow \infty} f(x) \right\} = \min \left\{ 1, \frac{128}{125} \right\} = 1$$

for all $x > 1/\sqrt{5}$. Therefore, inequality (1) holds for all $x > 0$.

Replacing x in (1) by $\sqrt[3]{a/b}$, $\sqrt[3]{b/c}$, and $\sqrt[3]{c/a}$ in turn yields:

$$\frac{a\sqrt[3]{2}}{\sqrt[3]{a+b}} \geq \frac{5a^{\frac{2}{3}} - b^{\frac{2}{3}}}{4}, \quad \frac{b\sqrt[3]{2}}{\sqrt[3]{b+c}} \geq \frac{5b^{\frac{2}{3}} - c^{\frac{2}{3}}}{4}, \quad \frac{c\sqrt[3]{2}}{\sqrt[3]{c+a}} \geq \frac{5c^{\frac{2}{3}} - a^{\frac{2}{3}}}{4}.$$

Adding these inequalities produces the desired result. Equality holds if and only if $a = b = c$.

Solution to part (c) by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

If we set $a = x^2$, $b = y^2$, and $c = z^2$, the proposed inequality becomes

$$\sum_{\text{cyclic}} \frac{x^2}{\sqrt{x^2 + y^2}} \geq \frac{x + y + z}{\sqrt{2}}.$$

By squaring both sides, we can rewrite the inequality as

$$\sum_{\text{cyclic}} \frac{x^4}{x^2 + y^2} + 2 \sum_{\text{cyclic}} \frac{x^2 y^2}{\sqrt{(x^2 + y^2)(y^2 + z^2)}} \geq \frac{(x + y + z)^2}{2}.$$

By the Rearrangement Inequality,

$$\begin{aligned} \sum_{\text{cyclic}} \frac{x^2 y^2}{\sqrt{(x^2 + y^2)(y^2 + z^2)}} &= \sum_{\text{cyclic}} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \cdot \frac{1}{\sqrt{y^2 + z^2}} \\ &\geq \sum_{\text{cyclic}} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \cdot \frac{1}{\sqrt{x^2 + y^2}} \\ &= \sum_{\text{cyclic}} \frac{x^2 y^2}{x^2 + y^2}. \end{aligned}$$

Thus, it suffices to prove that

$$\sum_{\text{cyclic}} \frac{x^4}{x^2 + y^2} + 2 \sum_{\text{cyclic}} \frac{x^2 y^2}{x^2 + y^2} \geq \frac{(x + y + z)^2}{2}. \quad (2)$$

Moreover, we observe that

$$\sum_{\text{cyclic}} \frac{x^4 - y^4}{x^2 + y^2} = \sum_{\text{cyclic}} (x^2 - y^2) = 0.$$

Hence,

$$\sum_{\text{cyclic}} \frac{x^4}{x^2 + y^2} = \frac{1}{2} \sum_{\text{cyclic}} \frac{x^4 + y^4}{x^2 + y^2}.$$

Therefore, inequality (2) is successively equivalent to

$$\begin{aligned} \sum_{\text{cyclic}} \frac{x^4 + y^4}{x^2 + y^2} + \sum_{\text{cyclic}} \frac{4x^2 y^2}{x^2 + y^2} &\geq (x + y + z)^2, \\ \sum_{\text{cyclic}} \frac{x^2 + y^2}{2} + \sum_{\text{cyclic}} \frac{2x^2 y^2}{x^2 + y^2} &\geq 2 \sum_{\text{cyclic}} xy, \\ \sum_{\text{cyclic}} \frac{(x - y)^4}{2(x^2 + y^2)} &\geq 0, \end{aligned}$$

which is clearly true. Therefore, the inequality holds. Equality holds if and only if $a = b = c$.

All three parts were also solved by SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and VO QUOC BA CAN. Part (a) was solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. Part (c) was also solved by CAO MINH QUANG. The proposer solved parts (a) and (b).

There were several solvers who had submitted correct solutions to the original 3182, which was the same as 3096 [2005 : 544, 547; 2006 : 531]. Most of them were already listed there as having solved 3096. However, CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and D. KIPP JOHNSON, Beaverton, OR, USA should be added to that list.

3183. [2006 : 463, 464] Proposed by Arkady Alt, San Jose, CA, USA.

Let ABC be a triangle with inradius r and circumradius R . If s is the semiperimeter of the triangle, prove that

$$\sqrt{3}s \leq r + 4R.$$

Remark by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and D. Kipp Johnson, Beaverton, OR, USA.

This is a very old problem. Its origin is referred back to the year 1872 in [1, item 5.5].

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3184. [2006 : 463] Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.

For any real number x , let (x) denote the fractional part of x ; that is, $(x) = x - [x]$, where $[x]$ is the greatest integer not exceeding x . Given $n \in \mathbb{Z}$, find all solutions of the equation

$$(x^2) - n(x) = 0.$$

Solution by Michel Bataille, Rouen, France.

[Ed: We will use $\{x\}$ instead of (x) to denote the fractional part of x .]

Let S_n denote the set of all real solutions to the equation

$$\{x^2\} - n\{x\} = 0. \tag{1}$$

Note that $\mathbb{Z} \subseteq S_n$, since $\{k^2\} = \{k\} = 0$ for all $k \in \mathbb{Z}$.

If $n < 0$ and a is not an integer, then $\{a^2\} \geq 0$ and $\{a\} > 0$, so that a cannot be a solution of (1). Thus, $S_n = \mathbb{Z}$ if $n < 0$.

If $n = 0$, then $S_n = S_0 = \{\pm\sqrt{m} : m \in \mathbb{Z}, m \geq 0\}$.

We will now determine the non-integer solutions of (1) for $n \geq 1$. Let s be such a solution, and let $k = \lfloor s \rfloor$. Then $\{s\} \neq 0$ and $n\{s\} = \{s^2\} \in [0, 1)$. Hence, $0 < n\{s\} < 1$ and $k < s < k + (1/n)$.

Case 1. $k \geq 0$.

Since $s > k \geq 0$, we have $s^2 > k^2$. Then $\lfloor s^2 \rfloor = k^2 + m$ for some integer $m \geq 0$, and

$$(k + \{s\})^2 = s^2 = k^2 + m + \{s^2\} = k^2 + m + n\{s\}.$$

Thus, we see that $\{s\}$ is a solution to the quadratic equation $p(x) = 0$, where $p(x) = x^2 + (2k - n)x - m$. Note that $m \neq 0$ (since $\{s\}$ is not an integer) and that $\{s\}$ is the unique positive solution to $p(x) = 0$, namely,

$$\{s\} = \frac{1}{2}(n - 2k + \sqrt{(2k - n)^2 + 4m}).$$

Then $s = \frac{1}{2}(n + \sqrt{(2k - n)^2 + 4m})$.

Since $0 < \{s\} < 1/n$, and since $p(0) = -m < 0$ and $p(\{s\}) = 0$, we must have $p(1/n) > 0$, which yields $m + 1 < (2kn + 1)/n^2$. Then $m + 1 \leq \lfloor 2k/n \rfloor$, because if $n > 1$, then

$$\left\lfloor \frac{2kn + 1}{n^2} \right\rfloor = \left\lfloor \frac{\lfloor 2k/n \rfloor + (1/n)}{n} \right\rfloor = \left\lfloor \frac{2k}{n} \right\rfloor,$$

and if $n = 1$, then $(2kn + 1)/n^2 = 2k + 1 = \lfloor 2k/n \rfloor + 1$. Therefore, $m \in \{1, 2, \dots, \lfloor 2k/n \rfloor - 1\}$. Then $\lfloor 2k/n \rfloor - 1 \geq 1$; whence, $k \geq n$.

Conversely, let $s = k + \alpha_m$, where k is an integer with $k \geq n$ and $\alpha_m = \frac{1}{2}(n - 2k + \sqrt{(2k - n)^2 + 4m})$ for $m \in \{1, 2, \dots, \lfloor 2k/n \rfloor - 1\}$. Then one can verify that $\alpha_m \in (0, 1/n)$. In addition,

$$s^2 = k^2 + \alpha_m^2 + 2k\alpha_m = k^2 + m + n\alpha_m;$$

whence, $\{s^2\} = n\alpha_m = n\{s\}$ and s is a solution of (1).

Thus, the non-integral positive solutions to (1) are the numbers of the form $\frac{1}{2}(n + \sqrt{(2k - n)^2 + 4m})$ for $k \geq n$ and $m \in \{1, 2, \dots, \lfloor 2k/n \rfloor - 1\}$.

Case 2. $k \leq -1$.

First we assume that $n > 1$. Since $k < s < k + (1/n)$, we have $s^2 = k^2 - m + \{s^2\}$, where $m \in \{1, 2, \dots, -\lfloor 2k/n \rfloor\}$. As in Case 1, $\{s\}$ must be a solution of the quadratic equation $x^2 + (2k - n)x + m = 0$. Since $0 < \{s\} < 1/n$, we must have $m < 1 - (2k/n) - (1/n^2)$, or (since $(2kn + 1)/n^2$ is not an integer),

$$m \leq 1 + \left\lfloor -\frac{2k + 1/n}{n} \right\rfloor = 1 + \left\lfloor \frac{-2k - 1}{n} \right\rfloor.$$

We note that $\{s\}$ is the smallest solution to the quadratic equation. Hence, $\{s\} = \frac{1}{2}(n - 2k - \sqrt{(n - 2k)^2 - 4m})$ and $s = \frac{1}{2}(n - \sqrt{(n - 2k)^2 - 4m})$. Conversely, it can be checked that such a number is a solution to (1).

Thus, if $n > 1$, the negative non-integer solutions of (1) are the real numbers $\frac{1}{2}(n - \sqrt{(n - 2k)^2 - 4m})$ for $m \in \{1, 2, \dots, 1 + \lfloor (-2k - 1)/n \rfloor\}$ and $k \leq -1$.

If $n = 1$, we find in a similar manner that the negative solutions of (1) are $\frac{1}{2}(1 - \sqrt{(1 - 2k)^2 - 4m})$, where $m \in \{1, 2, \dots, -2k - 1\}$ and $k \leq -1$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. There was one partly incorrect solution submitted.

3185. Replacement. [2007 : 40, 44] Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.

Let a_n denote the units digit of $(4n)^{(3n)^{(2n)^n}}$. Find all positive integers n such that $\sum_{i=1}^n a_i \geq 4n$.

Solution by Michel Bataille, Rouen, France.

We show that the given inequality holds except for $n = 5$.

Modulo 10, we have $4n \equiv 4$ when $n \equiv 1$ or $n \equiv 6$, $4n \equiv 8$ when $n \equiv 2$ or $n \equiv 7$, $4n \equiv 2$ when $n \equiv 3$ or $n \equiv 8$, $4n \equiv 6$ when $n \equiv 4$ or $n \equiv 9$, and $4n \equiv 0$ when $n \equiv 0$ or $n \equiv 5$. The powers of 4, 8, 2, 6, 0 are given modulo 10 as follows:

- $\{4^m\}_{m \geq 1}$ is the 2-periodic sequence 4, 6, ...
- $\{8^m\}_{m \geq 1}$ is the 4-periodic sequence 8, 4, 2, 6, ...
- $\{2^m\}_{m \geq 1}$ is the 4-periodic sequence 2, 4, 8, 6, ...
- $\{6^m\}_{m \geq 1}$ and $\{0^m\}_{m \geq 1}$ are constant sequences.

As a result, in order to compute a_n , we only need to determine the values of $(3n)^{(2n)^n}$ modulo 4. But, since $(2n)^n$ is always even, it is clear that $(3n)^{(2n)^n} \equiv 1 \pmod{4}$ or $(3n)^{(2n)^n} \equiv 0 \pmod{4}$ according as n is odd or even. It is then easy to obtain the values of a_n , $S_n = \sum_{i=1}^n a_i$, and $4n$ for $n = 1, 2, \dots, 15$. They are given in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	4	6	2	6	0	6	8	6	6	0	4	6	2	6	0
S_n	4	10	12	18	18	24	32	38	44	44	48	54	56	62	62
$4n$	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60

The sequence $\{a_n\}$ is 10-periodic and, as the table shows, the sum of a_i over 10 consecutive values of i is $6 + 8 + 6 + 6 + 0 + 4 + 6 + 2 + 6 + 0 = 44$. Therefore, for all integers k and m with $k \geq 0$ and $m > 0$, we see that $S_{m+10k} = S_m + 44k$. Moreover, from the table, we get $S_m \geq 4m$ for

$m \in \{6, 7, \dots, 15\}$. It follows that if $n \geq 6$, then, by writing $n = m + 10k$ with $k \geq 0$ and $6 \leq m \leq 15$, we have

$$S_n = S_{m+10k} = S_m + 44k \geq 4m + 44k \geq 4m + 40k = 4n.$$

We also see that $S_n \geq 4n$ for $n = 1, 2, 3, 4$ and $S_5 < 4 \times 5$. The result follows.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; and the proposer. There was one incorrect submission.

This was a replacement problem. There were a number of solvers who solved the originally posed 3185 [2006 : 463, 465], which was a repeat of 2935 [2004 : 174, 176; 2005 : 188]. The following solvers should be added to the list of solvers for 2935: MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and PANOS E. TSAOUSSOGLU, Athens, Greece.

3186. [2006 : 463, 465] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $f(x)$ be a function on an interval I which is convex for $x \geq a$ for some $a \in I$. Suppose that for all $x_1, x_2, \dots, x_n \in I$ which satisfy $x_1 + x_2 + \dots + x_n = na$, the following inequality holds:

$$\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right).$$

Prove that this same inequality holds for all $x_1, x_2, \dots, x_n \in I$ such that $x_1 + x_2 + \dots + x_n \geq na$.

Solution by the proposer.

Consider any $b \in I$ such that $b > a$. Let $x_1, x_2, \dots, x_n \in I$ such that $x_1 + x_2 + \dots + x_n = nb$. We have to prove that

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(b). \quad (1)$$

Without loss of generality, assume that $x_1 \leq x_2 \leq \dots \leq x_n$. If $x_1 \geq a$, then (1) is just Jensen's Inequality for convex functions. We now suppose that $x_1 < a$. Since $x_1 + x_2 + \dots + x_n = nb > na$, we must have $x_n > a$. There is some $k \in \{1, 2, \dots, n-1\}$ such that $x_{n-k} < a \leq x_{n-k+1}$. Define

$$d = \frac{x_{n-k+1} + \dots + x_n}{k} \quad \text{and} \quad c = \frac{na - (x_1 + \dots + x_{n-k})}{k}.$$

Then

$$k(d - c) = (x_1 + x_2 + \dots + x_n) - na = n(b - a), \quad (2)$$

which implies that $c < d$. Since $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_{n-k} < x_{n-k+1}$, we have

$$b = \frac{x_1 + x_2 + \cdots + x_n}{n} < \frac{x_{n-k+1} + \cdots + x_n}{k} = d.$$

Since $x_1 \leq x_2 \leq \cdots \leq x_{n-k} < a$, we have $\frac{x_1 + \cdots + x_{n-k}}{n-k} < a$ and hence $a < c$. Thus, $a < b < d$ and $a < c < d$. In other words, both b and c are in the interval (a, d) . Note that f is convex on $[a, d]$.

According to Jensen's Inequality for convex functions,

$$f(x_{n-k+1}) + \cdots + f(x_n) \geq kf(d). \quad (3)$$

On the other hand, by the given hypothesis, we have

$$f(x_1) + \cdots + f(x_{n-k}) + kf\left(\frac{na - (x_1 + \cdots + x_{n-k})}{k}\right) \geq nf(a);$$

that is,

$$f(x_1) + \cdots + f(x_{n-k}) + kf(c) \geq nf(a). \quad (4)$$

Adding (3) and (4), we get

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq k(f(d) - f(c)) + nf(a).$$

Thus, we can prove (1) by showing that $k(f(d) - f(c)) \geq n(f(b) - f(a))$. In view of (2), this is equivalent to

$$\frac{f(d) - f(c)}{d - c} \geq \frac{f(b) - f(a)}{b - a}. \quad (5)$$

To prove this inequality, we will prove that

$$\frac{f(d) - f(c)}{d - c} \geq \frac{f(d) - f(a)}{d - a} \geq \frac{f(b) - f(a)}{b - a}. \quad (6)$$

The left inequality in (6) reduces to

$$(d - c)f(a) + (c - a)f(d) \geq (d - a)f(c);$$

that is,

$$(d - c)f(a) + (c - a)f(d) \geq ((d - c) + (c - a))f\left(\frac{(d - c)a + (c - a)d}{(d - c) + (c - a)}\right),$$

which is true by Jensen's Inequality (applied to f on the interval $[a, d]$). The right inequality in (6) reduces to

$$(d - b)f(a) + (b - a)f(d) \geq (d - a)f(b);$$

that is,

$$(d - b)f(a) + (b - a)f(d) \geq ((d - b) + (b - a))f\left(\frac{(d - b)a + (b - a)d}{(d - b) + (b - a)}\right),$$

which is also true by Jensen's Inequality. Thus, (6) is true and the proof is complete.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain. One incomplete solution was submitted.

The featured proof above could have ended with inequality (5), since that inequality is a standard property of convex functions expressing the fact that their graphs have increasing slopes.

3187. [2006 : 463, 465] *Proposed by Michel Bataille, Rouen, France.*

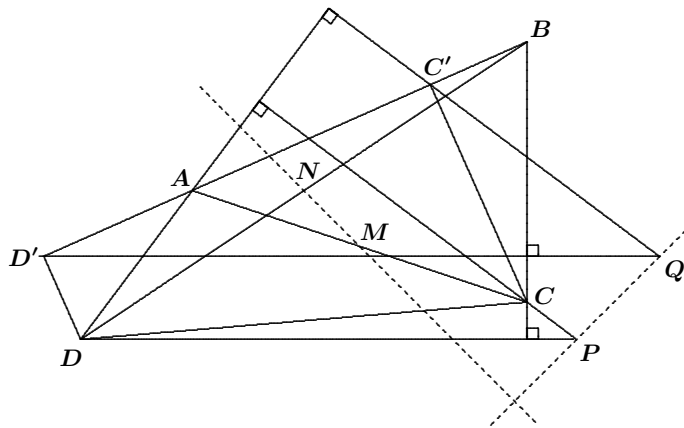
Let $ABCD$ be a planar quadrilateral which is not a parallelogram. Let C' and D' be the orthogonal projections onto the line AB of the points C and D , respectively. The perpendiculars from C to AD and from D to BC meet at P ; the perpendiculars from C' to AD and from D' to BC meet at Q . Show that PQ is perpendicular to the line through the mid-points of AC and BD .

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece, with minor modifications by the editor.

We make use of the known theorem:

For any four distinct points U, V, X , and Y , we have $XY \perp UV$ if and only if $XU^2 - XV^2 = YU^2 - YV^2$.

Let M and N be the mid-points of AC and BD , respectively.



Then $QP \perp MN$ if and only if $QN^2 - QM^2 = PN^2 - PM^2$. Using the Median Theorem for triangles QBD , QAC , PBD , and PAC , we see that this equation is equivalent to

$$\begin{aligned} & (QB^2 + QD^2 - \frac{1}{2} BD^2) - (QC^2 + QA^2 - \frac{1}{2} AC^2) \\ &= (PB^2 + PD^2 - \frac{1}{2} BD^2) - (PC^2 + PA^2 - \frac{1}{2} AC^2); \end{aligned}$$

that is,

$$(QB^2 - QC^2) + (QD^2 - QA^2) = (PB^2 - PC^2) + (PD^2 - PA^2).$$

Since $QD' \perp BC$, $QC' \perp AD$, $PD \perp BC$, and $PC \perp AD$, the above is equivalent to

$$(D'B^2 - D'C^2) + (C'D^2 - C'A^2) = (DB^2 - DC^2) + (CD^2 - CA^2);$$

that is,

$$C'D^2 - (DB^2 - D'B^2) = D'C^2 - (CA^2 - C'A^2).$$

Since $\triangle D'BD$ and $\triangle C'AC$ are right-angled, we can use the Pythagorean Theorem to rewrite the last equation as $C'D^2 - D'D^2 = D'C^2 - C'C^2$. Then, since $\triangle C'D'D$ and $\triangle C'D'C$ are right-angled, we obtain the equivalent equation $D'C'^2 = D'C'^2$, which is true.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Only Demis and the proposer used purely geometric techniques. The others solved the problem using trigonometry and/or vectors.

3188. Replacement. [2007 : 109, 112] Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let z_1, z_2, \dots, z_n be the zeroes of the complex polynomial

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

where $a_0 \neq 0$. Prove that

$$\det \begin{bmatrix} n & z_1 & z_2 & \cdots & z_n \\ z_1 & 1 + z_1^2 & 1 & \cdots & 1 \\ z_2 & 1 & 1 + z_2^2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n & 1 & 1 & \cdots & 1 + z_n^2 \end{bmatrix} = a_1^2.$$

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela, modified slightly by the editor.

Note that the matrix in the problem statement above is A_n^2 , where

$$A_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & z_1 & 0 & \cdots & 0 \\ 1 & 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & z_n \end{bmatrix}.$$

The determinant in the problem is $\det(A_n^2) = D_n^2$, where $D_n = \det(A_n)$. Expanding D_n along the last row, we find that, for $n \geq 2$,

$$D_n = (-1)^{n+2} \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ z_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_{n-1} & 0 \end{bmatrix} + (-1)^{2n+2} z_n D_{n-1}$$

$$= -z_1 z_2 \cdots z_{n-1} + z_n D_{n-1}.$$

Since $D_1 = -1$, an easy induction shows that

$$D_n = - \sum_{k=1}^n (z_1 z_2 \cdots \hat{z}_k \cdots z_n),$$

where \hat{z}_k indicates that the factor z_k is missing. Then $D_n = (-1)^n a_1$, since $a_1 = (-1)^{n-1} \sum_{k=1}^n (z_1 z_2 \cdots \hat{z}_k \cdots z_n)$ by Vieta's Formula. It follows that $\det(A_n^2) = a_1^2$.

Note that the condition $a_0 \neq 0$ is superfluous.

Also solved by MICHEL BATAILLE, Rouen, France; G.P. HENDERSON, Garden Hill, Campbellcroft, ON; and the proposer. There was one incomplete solution submitted.

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